

Lie Transformation Groups

An Introduction to Symmetry Group Analysis of Differential Equations

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Preface

These lecture notes cover the material of a four hour course on Lie transformation groups. The goal of this course is to provide the foundations of symmetry group analysis of differential equations, a vast and very active research area with connections to Lie group theory, differential geometry, differential equations, calculus of variations, integrable systems, and mathematical physics, among others. The standard reference in this field is P. J. Olver's seminal work [14], and to a large part this set of notes will closely follow his exposition. We do, however, take a slightly different approach to the foundations of the field and try to give a reasonably complete exposition of the underlying theory of local transformation groups, which in turn is based on H. Sussmann's study [18] of the integrability of distributions of non-constant rank. For connecting this material with Olver's approach, we have found V. Schmidt's doctoral thesis [17] very helpful.

I have tried to make these notes self-contained, pre-supposing only that the reader be familiar with the material covered in basic courses on Differential Geometry and Lie Groups. More precisely, the presentation is a direct continuation of my courses [9] and [11] (which in turn is mostly based on [3]), as well as the chapter on submanifolds in [10]. In particular, these notes supply proofs for many results that are left to the reader in [14]. Clearly what we can provide here is only a first step into the field, but it should allow the reader to delve into the subject, e.g., by further studying [14].

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Chapter 1

Lie transformation groups

1.1 Basic concepts

Throughout this course, by a manifold we mean a smooth (C^∞) manifold. However, we do not a priori assume further properties of the natural manifold topology like Hausdorff or second countable. In this chapter we will follow [3].

We begin by recalling [11, Def. 16.1]:

1.1.1 Definition. A transformation of a manifold M is a diffeomorphism from M onto M . A group G acts on M as a transformation group (on the left) if there exists a map $\Phi: G \times M \rightarrow M$ satisfying:

- (i) For every $g \in G$ the map $\Phi_g: m \mapsto \Phi(g, m)$ is a transformation of M .
- (ii) For all $g, h \in G$, $\Phi_g \circ \Phi_h = \Phi_{gh}$.

In particular, $\Phi_e = \text{id}_M$.

G acts effectively on M if $\Phi_g(m) = m$ for all $m \in M$ implies $g = e$. G acts freely on M if it has no fixed points, i.e., if $\Phi_g(m) = m$ for some $m \in M$ implies that $g = e$.

Our main object of study is introduced next.

1.1.2 Definition. A Lie group G acts as a Lie transformation group on a manifold M if there exists a smooth surjection $\Phi: G \times M \rightarrow M$ such that for all $g, h \in G$, $\Phi_g \circ \Phi_h = \Phi_{gh}$.

We first note that any Lie transformation group is a transformation group in the sense of 1.1.1. To see this, note first that Φ being surjective implies that $\Phi_e = \text{id}_M$. In fact, given $m \in M$ there exist $m' \in M$, $h \in G$ with $\Phi_h(m') = m$. Therefore, $\Phi_e(m) = \Phi_e \circ \Phi_h(m') = \Phi_h(m') = m$. Consequently, $\Phi_g^{-1} = \Phi_{g^{-1}}$ for all $g \in G$, so each Φ_g is a transformation of M , and we have verified all conditions from Def. 1.1.1. We will often briefly write gm instead of $\Phi_g(m)$.

1.1.3 Example. Let X be a complete vector field on a Hausdorff manifold M . Then the flow of X defines a smooth map $\Psi := \text{Fl}^X: \mathbb{R} \times M \rightarrow M$ with $\Psi(s, \Psi(t, m)) = \Psi(s + t, m)$ for all $s, t \in \mathbb{R}$ and all $m \in M$. Ψ therefore induces a Lie transformation group of the Lie group $(\mathbb{R}, +)$ on M . Each $\Psi_t := m \mapsto \Psi(t, m)$ is a transformation on M and $t \mapsto \Psi_t$ is a homomorphism of \mathbb{R} into the group of transformations on M .

1.1.4 Lemma. *Let G be a Lie group acting as a Lie transformation group on M and let H be a Lie subgroup of G . Then also H acts as a Lie transformation group on M .*

Proof. Let $j : H \hookrightarrow G$ be the natural injection and let $\Phi : G \times M \rightarrow M$ be as in 1.1.2. Then $\Phi_H : H \times M \rightarrow M$, $\Phi_H := \Phi \circ (j \times \text{id}_M)$ does the trick. \square

If G acts as a transformation group on M via Φ then a subset A of M is called *invariant* under G if $\Phi(G \times A) \subseteq A$.

1.1.5 Proposition. *If a regular submanifold M' of M is invariant under the action of a Lie transformation group G on M then G acts naturally on M' as a Lie transformation group.*

Proof. Let $j : G \times M' \hookrightarrow G \times M$ be the natural injection. Then since M' is regular, the map $\Phi' : G \times M' \rightarrow M'$ induced by $\Phi \circ j$ is smooth (see [10, 1.1.14]), hence it defines an action of G on M' . \square

1.1.6 Example. $GL(n, \mathbb{R})$ acts on \mathbb{R}^n as a Lie transformation group via $(A, x) \mapsto Ax$. By 1.1.4, so does $O(n, \mathbb{R})$. Since S^{n-1} is a regular submanifold of \mathbb{R}^n invariant under $O(n, \mathbb{R})$, 1.1.5 implies that $O(n, \mathbb{R})$ acts on S^{n-1} as a Lie transformation group.

For the notion of quotient manifold, cf. [11, Sec. 15]. We say that a transformation group G *preserves* an equivalence relation ρ on M if $(m_1, m_2) \in \rho$ and $g \in G$ imply $(gm_1, gm_2) \in \rho$.

1.1.7 Proposition. *If M/ρ is a quotient manifold of M and if the equivalence relation ρ is preserved by a Lie transformation group G on M then G acts naturally on M/ρ as a Lie transformation group.*

Proof. Denote by $\pi : M \rightarrow M/\rho$ the natural surjection. Then $\text{id} \times \pi : G \times M \rightarrow G \times (M/\rho)$ is a submersion, so $G \times (M/\rho)$ is a quotient manifold of $G \times M$. Moreover, the smooth map $\pi \circ \Phi : G \times M \rightarrow M/\rho$ is an invariant of the corresponding equivalence relation on $G \times M$ since $(g, m_1) \sim (g, m_2)$ if and only if $(gm_1, gm_2) \in \rho$, which implies $\pi \circ \Phi(g, m_1) = \pi(gm_1) = \pi(gm_2) = \pi \circ \Phi(g, m_2)$. By [11, 15.13] it therefore projects to a smooth map

$$\Phi_\rho : G \times (M/\rho) \rightarrow M/\rho, \quad (g, \pi(m)) \mapsto \pi(gm)$$

which defines the required action of G on M/ρ since

$$\begin{aligned} \Phi_\rho(g_1, \Phi_\rho(g_2, \pi(m))) &= \Phi_\rho(g_1, \pi \circ \Phi(g_2, m)) = \pi \circ \Phi(g_1, \Phi(g_2, m)) \\ &= \pi \circ \Phi(g_1 g_2, m) = \Phi_\rho(g_1 g_2, \pi(m)). \end{aligned}$$

\square

1.1.8 Example. Any Lie group G acts on itself as a Lie transformation group via multiplication $\mu : G \times G \rightarrow G$. A subgroup H of G determines an equivalence relation ρ on G by

$$(g_1, g_2) \in \rho :\Leftrightarrow \exists h \in H : g_1 = g_2 h$$

and this relation is preserved by the transformation group. If the set of orbits G/H (i.e., the set of left cosets) is a quotient manifold of G (cf. [11, Sec. 15, 20]) then G acts naturally on G/H as a Lie transformation group with action Φ given by $(g, aH) \mapsto (ga)H$. A sufficient condition for G/H to be a quotient manifold of G is that H is closed, not open, and connected (see [11, 20.5]).

1.1.9 Proposition. *If H is a closed subgroup of a Lie group G such that G/H is a quotient manifold of G , then G/H is a Hausdorff manifold.*

Proof. Recall from [11, 15.9] that the natural manifold topology of G/H is the quotient topology with respect to the natural projection $\pi : G \rightarrow G/H$. Suppose first that $gH \in G/H$ is distinct from H . Then gH is a closed subset of G that does not contain e . Hence there exists a neighborhood U of e in G which is disjoint from gH . Pick a neighborhood V of e in G such that $V^{-1}V \subseteq U$. We show that $\pi(Vg)$ and $\pi(V)$ are disjoint neighborhoods of gH and H , respectively. Indeed, if $\pi(Vg) \cap \pi(V) \neq \emptyset$ then there exist $a, b \in V$ such that $agH = bH$, so $a^{-1}b \in gH$. But $a^{-1}b \in V^{-1}V \subseteq U$, so we obtain a contradiction.

In general, if gH and $g'H$ are distinct elements of G/H then so are H and $(g^{-1}g')H$ and by what we have shown above they possess disjoint neighborhoods W and W' . Denoting by ϕ the action of G on G/H from 1.1.8, it follows that $\Phi_g(W)$ and $\Phi_g(W')$ are disjoint neighborhoods of gH and $g'H$. \square

Recall from [11, Sec. 4, 6] that a vector field $X \in \mathfrak{X}(G)$ is called left- (resp. right)-invariant if $L_g^*X = X$ (resp. $R_g^*X = X$) for each $g \in G$. We denote by $\mathfrak{X}_L(G)$ (resp. $\mathfrak{X}_R(G)$) the space of left- (resp. right)-invariant vector fields on G . Also, we identify $\mathfrak{X}_L(G)$ with the Lie algebra \mathfrak{g} via $T_eG \ni v \mapsto L^v \in \mathfrak{X}_L(G)$, where $L^v(g) := T_eL_g(v)$. The inverse of this map is $X \mapsto X(e)$.

Similarly, \mathfrak{g} is isomorphic to $\mathfrak{X}_R(G)$ via $T_eG \ni v \mapsto R^v \in \mathfrak{X}_R(G)$ with $R^v(g) := T_eR_g(v)$. We usually equip \mathfrak{g} with the Lie algebra structure induced by $[v, w] := [L^v, L^w](e)$. Alternatively, we may set $[v, w]_R := [R^v, R^w](e)$. The resulting Lie algebra will be denoted by \mathfrak{g}_R . We have:

1.1.10 Proposition. *Let G be a Lie group. Then*

(i) *The map $F : L^v \mapsto -R^v$ is a Lie algebra isomorphism from $\mathfrak{X}_L(G)$ onto $\mathfrak{X}_R(G)$.*

(ii) *The map $f : v \mapsto -v$ is a Lie algebra isomorphism from \mathfrak{g} onto \mathfrak{g}_R .*

Proof. (i) By what was said above, F is a linear isomorphism, since both $L^v \mapsto v \in \mathfrak{g}$ and $v \mapsto R^v$ are. Furthermore, by [11, 6.1 (iii)] we have:

$$[F(L^v), F(L^w)] = [R^v, R^w] = R^{-[v, w]} = -R^{[v, w]} = F(L^{[v, w]}) = F([L^v, L^w]),$$

so F is a Lie algebra isomorphism.

(ii) We have $[v, w]_R = [R^v, R^w](e) = R^{-[v, w]}(e) = -[v, w]$. Hence

$$[f(v), f(w)]_R = [-v, -w]_R = [v, w]_R = -[v, w] = f([v, w]).$$

\square

1.1.11 Remark. By [11, 6.1] The map F from 1.1.10 can also be written as $X \mapsto \nu^*X$. In fact, if $X \in \mathfrak{X}_L(G)$ then since $\nu \circ R_g = L_{g^{-1}} \circ \nu$ we have

$$R_g^*(\nu^*X) = (\nu \circ R_g)^*X = (L_{g^{-1}} \circ \nu)^*X = \nu^*(L_{g^{-1}}^*X) = \nu^*X,$$

so $\nu^*X \in \mathfrak{X}_R(G)$, and by symmetry $\nu^* : \mathfrak{X}_L(G) \rightarrow \mathfrak{X}_R(G)$ is an isomorphism. Moreover, $\nu^*(L^v)(e) = (T_e\nu)^{-1} \circ L^v \circ \nu(e) = -v$, so $\nu^*(L^v) = R^{-v} = F(L^v)$.

Let now $\Phi: G \times M \rightarrow M$ be a Lie transformation group and let $m \in M$. Then the map

$$\begin{aligned}\Phi_m &: G \rightarrow M \\ \Phi_m(g) &:= \Phi(g, m)\end{aligned}\tag{1.1.1}$$

is smooth. Given any $v \in T_e G = \mathfrak{g}$, let

$$\begin{aligned}\Phi(v) &: M \rightarrow TM \\ \Phi(v)(m) &:= T_e \Phi_m(v)\end{aligned}\tag{1.1.2}$$

Then $\Phi(v)(m) = T\Phi(v, 0_m)$, so $\Phi(v)$ is smooth. Also, with $\pi: TM \rightarrow M$ and $\tilde{\pi}: T(G \times M) \rightarrow G \times M$ the canonical projections, $\pi \circ \Phi(v)(m) = \Phi(\tilde{\pi}(v, 0_m)) = \Phi(e, m) = m$, so $\Phi(v)$ is a smooth section of TM , i.e., $\Phi(v) \in \mathfrak{X}(M)$. We set

$$\mathcal{R}(G, M) := \{\Phi(v) \mid v \in \mathfrak{g}\}.\tag{1.1.3}$$

$\mathcal{R}(G, M)$ is called the *Killing algebra* of Φ . Since $v \mapsto \Phi(v)$ is linear, it is a finite-dimensional vector space. In fact, we even have:

1.1.12 Proposition. *The map $\Phi: v \mapsto \Phi(v)$ is a Lie algebra homomorphism from \mathfrak{g}_R onto $\mathcal{R}(G, M)$.*

Proof. Given any $v \in \mathfrak{g}$, we first show that for any $m \in M$, the right-invariant vector field Y on G with $Y(e) = v$ is Φ_m -related to $\Phi(v)$. In fact, noting that $\Phi_m \circ R_g(h) = \Phi_m(hg) = \Phi(h, \Phi_m(g)) = \Phi_{m'}(h)$ with $m' := \Phi_m(g)$, we obtain:

$$T_g \Phi_m(Y(g)) = T_g \Phi_m(T_e R_g(v)) = T_e(\Phi_m \circ R_g)(v) = T_e \Phi_{m'}(v) = \Phi(v)(\Phi_m(g)).$$

Thus for any $v \in \mathfrak{g}$,

$$T_g \Phi_m(Y(g)) = \Phi(v)(\Phi_m(g)).\tag{1.1.4}$$

From this, by [11, 4.4] we conclude that for $Z = R^w$ another right-invariant vector field we have

$$[\Phi(v), \Phi(w)] \circ \Phi_m = T\Phi_m \circ [Y, Z].$$

Inserting e , this gives

$$[\Phi(v), \Phi(w)]_m = T_e \Phi_m([Y, Z]_e) = T_e \Phi_m([v, w]_R)$$

(where we used that $[R^v, R^w] = R^{[v, w]_R}$, cf. 1.1.10). Thus, finally, $[\Phi(v), \Phi(w)] = \Phi([v, w]_R)$, as claimed. \square

In the following result, we will start denoting the time variable in the flow map of a vector field with ε , for the sake of compatibility with the case of symmetry groups of differential equations later on.

1.1.13 Proposition. *If M is T_2 , then every vector field $\Phi(v) \in \mathcal{R}(G, M)$ is complete, with flow*

$$\text{Fl}_\varepsilon^{\Phi(v)}(m) = \Phi(\exp(\varepsilon v), m) = \exp(\varepsilon v) \cdot m.$$

In particular,

$$\Phi(v)(m) = \left. \frac{d}{d\varepsilon} \right|_0 \Phi(\exp(\varepsilon v), m).\tag{1.1.5}$$

Proof. In the proof of 1.1.12 we have seen that R^v is Φ_m -related to $\Phi(v)$. Thus by [11, 7.2] it follows that $\Phi_m \circ \text{Fl}_\varepsilon^{R^v} = \text{Fl}_\varepsilon^{\Phi(v)} \circ \Phi_m$. Moreover, by [11, 8.2], $\text{Fl}_\varepsilon^{R^v}(g) = \exp(\varepsilon v) \cdot g$. Therefore,

$$\text{Fl}_\varepsilon^{\Phi(v)}(m) = \text{Fl}_\varepsilon^{\Phi(v)} \circ \Phi_m(e) = \Phi_m(\exp(\varepsilon v) \cdot e) = \Phi(\exp(\varepsilon v), m).$$

Completeness of $\Phi(v)$ is immediate from this description. \square

1.1.14 Example. Let $\Phi: \mathbb{R} \times M \rightarrow M$ be a Lie transformation group with Lie group $(\mathbb{R}, +)$ and T_2 -manifold M . Then $\mathcal{R}(\mathbb{R}, M)$ is generated by $\Phi(\frac{d}{d\varepsilon}|_0)$. Moreover, the flow of $\Phi(\frac{d}{d\varepsilon}|_0)$ is given by Φ itself: $\text{Fl}_\varepsilon^{\Phi(\frac{d}{d\varepsilon}|_0)}(m) = \Phi(\varepsilon, m)$: this is immediate from 1.1.13 since $\exp(\varepsilon \frac{d}{d\varepsilon}|_0) = \varepsilon$.

Recall from [11, Sec. 16] that, given a Lie transformation group $\Phi: G \times M \rightarrow M$, we set

$$K := \{k \in G \mid \Phi_k = \text{id}_M\}.$$

Then K is a normal subgroup of G and [11, 16.3] demonstrates that the quotient group G/K acts effectively on M . Moreover, K is closed in G since we can write $K = \bigcap_{m \in M} \Phi_m^{-1}(m)$. Hence by [11, 21.6], if K is not open then the quotient manifold G/K is a Lie group. The exceptional case here is not very interesting though:

1.1.15 Remark. If K is open, then $\mathcal{R}(G, M) = 0$. Indeed, if K is open (and closed) then since $e \in K$ we must have $G_e \subseteq K$. Then because for all $v \in \mathfrak{g}$, $\exp(\varepsilon v) \in G_e$ it follows that $\Phi_m(\exp(\varepsilon v)) = m$ for all $m \in M$, and differentiating with respect to ε at $\varepsilon = 0$ implies that $\Phi(v)(m) = T_e \Phi_m(v) = 0$ for all m . Thus $\Phi(v) = 0$ and thereby $\mathcal{R}(G, M) = 0$.

1.1.16 Proposition. If K is not open then G/K acts effectively on M as a Lie transformation group and $\mathcal{R}(G/K, M) = \mathcal{R}(G, M)$.

Proof. Let

$$\begin{aligned} \Phi' : (G/K) \times M &\rightarrow M \\ (gK, m) &\mapsto \Phi(g, m) \end{aligned}$$

To see that Φ' is well-defined and smooth let $\pi: G \rightarrow G/K$ be the natural projection. Then $\pi \times \text{id}_M: G \times M \rightarrow (G/K) \times M$ is a submersion (cf. [11, 20.5]) and Φ is an invariant of the corresponding equivalence relation on $G \times M$: if $(g_1, m_1) \sim (g_2, m_2)$ then $g_1 K = g_2 K$ and $m_1 = m_2$, so $\Phi(g_1, m_1) = \Phi'(g_1 K, m_1) = \Phi'(g_2 K, m_2) = \Phi(g_2, m_2)$. Thus by [11, 15.13], the map Φ' , being the projection of Φ , is itself smooth. That Φ' gives an effective action has been shown in [11, 16.3].

Finally, for any $m \in M$ we have $\Phi_m = \Phi'_m \circ \pi$, so for any $v \in \mathfrak{g}$ we get $T_e \Phi_m(v) = T_e \Phi'_m \circ T_e \pi(v)$, i.e., $\Phi(v) = \Phi'(T_e \pi(v))$. Since $T_e \pi$ is a surjection, this gives $\mathcal{R}(G/K, M) = \mathcal{R}(G, M)$. \square

1.1.17 Proposition. Let Φ be a transformation group that acts effectively on a T_2 -manifold M . Then the map

$$\Phi: \mathfrak{g}_R \rightarrow \mathcal{R}(G, M)$$

defined by (1.1.2) is a Lie algebra isomorphism.

Proof. Using 1.1.12 it only remains to show that Φ is injective. Thus let $\Phi(v) = 0$. Then by 1.1.13 it follows that $\exp(\varepsilon v) \cdot m = m$ for all $m \in M$ and all $\varepsilon \in \mathbb{R}$. Since G acts effectively we conclude that $\exp(\varepsilon v) = e$ for all ε , giving $v = 0$. \square

1.1.18 Example. Let $G = GL(n, \mathbb{R})$ and $M = \mathbb{R}^n$ with $\Phi: (A, x) \mapsto A \cdot x$. To determine the map (1.1.2) we let A^{ij} be the standard coordinates on $GL(n, \mathbb{R})$ and x^k those on \mathbb{R}^n . Then

$$\begin{aligned} T_e \Phi_x \left(\frac{\partial}{\partial A^{ij}} \right) &= \sum_k \left(\frac{\partial}{\partial A^{ij}} (x^k \circ \Phi_x) \right) \left(\frac{\partial}{\partial x^k} \Big|_x \right) \\ &= \sum_{k,r} \left(\frac{\partial}{\partial A^{ij}} (A^{kr} x_r) \right) \frac{\partial}{\partial x^k} \Big|_x = x_j \frac{\partial}{\partial x^i} \Big|_x, \end{aligned}$$

so $\Phi = [B^{ij}] \mapsto \sum_{i,j} B^{ij} x_j \frac{\partial}{\partial x^i}$. Since G act effectively, Φ is an isomorphism.

1.1.19 Remark. There is a strong converse of 1.1.17, due to R. Palais (see [15, p. 95]): *If \mathcal{A} is a finite-dimensional Lie algebra of complete vector fields on a T_2 -manifold M , then there exists a connected Lie group G which acts effectively on M as a Lie transformation group and such that $\mathcal{A} = \mathcal{R}(G, M)$.*

1.2 Orbits under a Lie transformation group

From [11, Sec. 16] we know that any transformation group Φ acting on a manifold M induces an equivalence relation on M : $m \sim m'$ iff there exists some $g \in G$ with $m' = gm$. The equivalence class of any $x \in M$ is called the *orbit* of m under G . It is the range of the map $\Phi_m = g \mapsto \Phi(g, m)$, $G \rightarrow M$.

1.2.1 Definition. Let $\Phi: G \times M \rightarrow M$ be a transformation group on a manifold M . For any $m \in M$, the subgroup $G_m := \Phi_m^{-1}(m) = \{g \in G \mid g \cdot m = m\}$ is called the *isotropy group* of m .

1.2.2 Remark. (i) Isotropy groups at equivalent points in M are conjugate subgroups of G . In fact, suppose that $m' = gm$. Then $(gG_m g^{-1})m' = m'$, so $gG_m g^{-1} \subseteq G_{m'}$, and analogously $g^{-1}G_{m'}g \subseteq G_m$, so $gG_m g^{-1} = G_{m'}$.

(ii) For any $m \in M$, the map $\Phi_m: G \rightarrow M$ projects to a map $\Psi_m: G/G_m \rightarrow M$ defined by $gG_m \mapsto gm$. The range of Ψ_m is the orbit of m . Also, Ψ_m is injective: if $\Psi_m(g_1 G_m) = \Psi_m(g_2 G_m)$ then $g_1 m = g_2 m$, so $g_1^{-1}g_2 \in G_m$, i.e., $g_1 G_m = g_2 G_m$.

1.2.3 Remark. Let $\Phi: G \times M \rightarrow M$ be a Lie transformation group on a T_2 -manifold M . Then any G_m is a closed subgroup of G . Hence by [11, 21.7], G_m is either discrete or it admits a unique structure as a (regular) submanifold of G . In the latter case it is also a Lie subgroup of G . If $m' = gm$ then by 1.2.2 (i), G_m is mapped onto $G_{m'}$ by the diffeomorphism $L_g \circ R_{g^{-1}}$. Thus the isotropy groups at points of an orbit are either all discrete or are regular submanifolds and Lie subgroups of G that are pairwise diffeomorphic (since $G_m, G_{m'}$ are regular submanifolds, the restriction of $L_g \circ R_{g^{-1}}$ is also a diffeomorphism from G_m onto $G_{m'}$).

If G_m is open then it must be a union of connected components of G , which, by [11, 2.4] are precisely the cosets of the normal subgroup G_e . If, for example, $G = g_1 G_e \cup \dots \cup g_k G_e$ and $G_m = g_1 G_e \cup \dots \cup g_l G_e$, then $G/G_m = \{g_1 G_m, \dots, g_k G_m\}$, since for $g \in g_j G_e$ we have

$$\pi(g) = g \cdot \bigcup_{i=1}^l g_i G_e = \bigcup_{i=1}^l g G_e g_i = \bigcup_{i=1}^l g_j G_e g_i = g_j G_m.$$

Also, the orbit of m only consists of finitely many points (namely $\Psi_m(G/G_m) = \{g_1 \cdot m, \dots, g_k \cdot m\}$). Otherwise, we have:

1.2.4 Theorem. *Let $\Phi: G \times M \rightarrow M$ be a transformation group on a T_2 -manifold M and let $m \in M$. If the isotropy group G_m of m is not open in G then the map Ψ_m from 1.2.2 (ii) is an injective immersion of the quotient manifold G/G_m into M .*

Proof. Since $\Phi_m = \Psi_m \circ \pi$ (with $\pi: G \rightarrow G/G_m$), Ψ_m is the projection of the smooth map Φ_m , hence is itself smooth (see [10, 1.1.9]). Also, G/G_m is a quotient manifold of G by [11, 21.5]. Since Ψ_m is injective by 1.2.2 (ii), it remains to show that its rank in any point equals the dimension of G/G_m . Since π is a submersion, this is the case if and only if the rank of Φ_m is everywhere equal to $\dim(G/G_m)$. We begin by showing that this is true at e .

Let $v \in T_e G$ such that $T_e \Phi_m(v) = 0$. By (1.1.2) this means that the vector field $\Phi(v)$ has a zero at m . Therefore 1.1.13 implies that $\text{Fl}_\varepsilon^{\Phi(v)}(m) = \exp(\varepsilon v)m = m$ for all ε , i.e., $\exp(\varepsilon v) \in G_m$ for all ε . Now if G_m is discrete then the image of $\varepsilon \rightarrow \exp(\varepsilon v)$, being connected, must consist solely of $e \in G$, so $v = \frac{d}{d\varepsilon}|_0 \exp(\varepsilon v) = 0$. In this case, then, $T_e \Phi_m$ is injective, and so the rank of $T_e \Phi_m$ equals the dimension of G , and thereby the dimension of G/G_m .

If G_m is non-discrete then by 1.2.3 it is a regular submanifold of G . Hence $\varepsilon \mapsto \exp(\varepsilon v)$ is smooth as a map into G_m (see [10, 1.1.14]), and so $v = \frac{d}{d\varepsilon}|_0 \exp(\varepsilon v) \in T_e G_m$. Altogether, we obtain that $\ker(T_e \Phi_m) \subseteq T_e G_m$. Conversely, Φ_m is constant on G_m , so $T_e \Phi_m|_{T_e G_m} \equiv 0$, hence in fact $\ker(T_e \Phi_m) = T_e G_m$. Consequently, using [11, 21.7] we obtain

$$\text{rk}(T_e \Phi_m) = \dim G - \dim G_m = \dim G/G_m.$$

Finally, if g is an arbitrary point in G then $\Phi_m \circ R_g = \Phi_{m'}$, where $m' = gm$. Then since R_g is a diffeomorphism we have

$$\text{rk}_g(\Phi_m) = \text{rk}_e(\Phi_{m'}) = \dim G/G_{m'}.$$

Now by 1.2.3 G_m and $G_{m'}$ are either both discrete or they are diffeomorphic, so we conclude that the rank of Φ_m equals $\dim G/G_m$ for every $m \in M$. \square

1.2.5 Corollary. *Under the assumptions of 1.2.4, the orbit $G \cdot m$ of any $m \in M$ can be endowed with the structure of an immersive submanifold of M diffeomorphic to G/G_m .*

Proof. For clarity, we write $\tilde{\Psi}_m$ for Ψ_m , viewed as a (bijective) map from G/G_m to $G \cdot m$. Declaring $\tilde{\Psi}_m$ to be a diffeomorphism provides $G \cdot m$ with a differentiable structure with respect to which the inclusion map $j: G \cdot m \hookrightarrow M$ is an immersion since $j \circ \tilde{\Psi}_m = \Psi_m: G/G_m \rightarrow M$ is an immersion. \square

1.2.6 Remark. Suppose that G is connected and let $m \in M$. By 1.2.4 the orbit $G \cdot m$ can be discrete only if G_m is open. In this case, G_m is open and closed in G , so $G_m = G$ and therefore $G \cdot m = \{m\}$.

1.2.7 Example. Let $M = \mathbb{R}^2$, $G = (\mathbb{R}, +)$, and $X = -y\partial_x + x\partial_y$ the rotation vector field on M . Then we obtain a Lie transformation group Φ by

$$\Phi: (\varepsilon, (x, y)) \mapsto \text{Fl}_\varepsilon^X(x, y) = (x \cos \varepsilon - y \sin \varepsilon, x \sin \varepsilon + y \cos \varepsilon).$$

The orbits of Φ are the regular submanifolds $\{x^2 + y^2 = \text{const}\}$ and the singleton $\{(0, 0)\}$.

Recall that G is said to act transitively on M if for any $m, m' \in M$ there exists some $g \in G$ with $gm = m'$. Such a group action possesses only a single orbit, namely the manifold M itself. By 1.2.2 (i) this means that any Ψ_m is a bijection from G/G_m onto M . To further elaborate on this, we will need the following auxiliary result:

1.2.8 Lemma. *Let M^m and N^n be manifolds and suppose that M is second countable and that $m < n$. Then an immersion $\Psi : M \rightarrow N$ cannot be onto any open subset W of N .*

Proof. It suffices to take for W the domain of a chart χ in N . Let p be any point in $\Psi^{-1}(W)$ and choose charts $(U, \varphi = (x^1, \dots, x^m))$ around p in M and $(V, \eta = (y^1, \dots, y^n))$ around $\Psi(p)$ such that $\Psi(U) \subseteq V \subseteq W$ and such that $\eta \circ \Psi \circ \varphi^{-1} = (x^1, \dots, x^m) \rightarrow (x^1, \dots, x^m, 0, \dots, 0)$. Then $\eta(\Psi(U)) \subseteq \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$, so it has Lebesgue measure 0. It follows that also the image $\chi(\Psi(U))$ of this set under the smooth map $\chi \circ \eta^{-1}$ has Lebesgue measure 0.

As p varies in $\Psi^{-1}(W)$, the domains U cover the set $\Psi^{-1}(W)$. As M is second countable we may extract a countable subcover $\{U_k \mid k \in \mathbb{N}\}$ from this collection. Then the sets $\Psi(U_k)$ cover $W \cap \Psi(M)$, entailing

$$\chi(W \cap \Psi(M)) = \bigcup_{k \in \mathbb{N}} \chi(\Psi(U_k)).$$

But then $\chi(W \cap \Psi(M))$ has Lebesgue measure 0 and so it cannot be all of $\chi(W)$. It follows that $\Psi(M)$ cannot contain all of W . \square

Using this, we can prove:

1.2.9 Proposition. *Let G be a second countable Lie group that acts transitively as a Lie transformation group on the T_2 -manifold M . Then M is diffeomorphic to G/G_m , for any $m \in M$.*

Proof. Fix any $m \in G$. If G_m is open, then so is any gG_m , hence any point in G/G_m (because $\pi^{-1}(\pi(g)) = gG_m$), which is therefore discrete (cf. [11, Rem. 20.1]). Otherwise, by [11, 21.5] G/G_m possesses a differentiable structure as a quotient manifold of G . In both cases, the quotient map $\pi : G \rightarrow G/G_m$ is open and continuous, so also the topology of G/G_m is second countable. Hence if G/G_m were discrete it would be countable. But Ψ_m is bijective, so this would imply that M was countable, which is impossible. Hence G/G_m is not discrete, and so it is a quotient manifold of G with a countable basis for its topology. Also, by 1.2.4 Ψ_m is an injective immersion of G/G_m into M . Hence $\dim G/G_m \leq \dim M$. Since Ψ_m is onto M , 1.2.8 implies that the dimensions in fact are equal. As $\Psi_m : G/G_m \rightarrow M$ is an immersion, it follows that its tangent map is bijective at any point. Thus by the inverse function theorem it is a local diffeomorphism, hence a global diffeomorphism since it is bijective. \square

1.2.10 Corollary. *Let G be a second countable Lie group that acts transitively and freely as a Lie transformation group on the T_2 -manifold M . Then M is diffeomorphic to G .*

Proof. If G acts freely then $G_m = \{e\}$ for every $m \in M$. Hence $G/G_m = G$ and the result follows from 1.2.9. \square

1.2.11 Corollary. *Let G be a compact Lie group that acts transitively as a Lie transformation group on the T_2 -manifold M . Then M is compact.*

Proof. Since G is compact, it is second countable. By 1.2.9, M is diffeomorphic to G/G_m , for any $m \in M$. Let $\pi : G \rightarrow G/G_m$ be the quotient map. Then $\pi(G) = G/G_m$ is compact, hence so is M . \square

1.2.12 Example. We continue our study of the action of $O(n, \mathbb{R})$ on S^{n-1} as a Lie transformation group from 1.1.6. This action is given by

$$\begin{aligned}\Phi: O(n, \mathbb{R}) \times S^{n-1} &\rightarrow S^{n-1} \\ (A, x) &\mapsto Ax.\end{aligned}$$

For any vector $v \in S^{n-1}$ we can find an orthogonal matrix T with v as its first column, so $Te_1 = v$. Hence Φ has only a single orbit, S^{n-1} , i.e., it is transitive. The isotropy group G_{e_1} of e_1 , i.e., those elements of $O(n, \mathbb{R})$ that leave e_1 unchanged, are the matrices of the form

$$A = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$$

with $D \in O(n-1, \mathbb{R})$. By 1.2.9 we conclude that S^{n-1} is diffeomorphic to the quotient manifold $O(n, \mathbb{R})/G_{e_1}$, i.e., $S^{n-1} \cong O(n, \mathbb{R})/O(n-1, \mathbb{R})$.

1.3 Groups of transformations on a T_2 -manifold

1.3.1 Definition. A set G of transformations on a T_2 -manifold M which is a group under composition $(g_1, g_2) \mapsto g_1 \circ g_2$ ($g_1, g_2 \in G$) is called a group of transformations of M .

According to 1.1.1, the map $\Phi: (g, m) \mapsto g(m) \equiv gm$ then defines a transformation group on M . Moreover, this action obviously is effective. In this section we want to analyze whether G can be endowed with a Lie group structure such that Φ is smooth. The following example shows that such a structure may not be unique in general:

1.3.2 Example. Let G be the group of translations on \mathbb{R}^2 and let $\phi_a := z \mapsto z + a$. Then G is bijectively mapped onto \mathbb{R}^2 by $\phi_a \mapsto a$, and this defines a \mathcal{C}^∞ -structure on G , for any given \mathcal{C}^∞ -structure on \mathbb{R}^2 . But on \mathbb{R}^2 there are different such structures, e.g., the standard one and the one from [11, Ex. 14.3]. Both of these induce corresponding structures on G such that Φ is smooth.

Recall from 1.1.3 that a complete vector field X on M induces an action of $(\mathbb{R}, +)$ on M as a Lie transformation group via the flow of X , $\Psi_\varepsilon := m \mapsto \text{Fl}_\varepsilon^X(m)$. We call the vector field X *tangent* to G if all the resulting transformations Ψ_ε belong to G .

1.3.3 Definition. A group of transformations on a T_2 -manifold M is called a Lie group of transformations if it admits a Lie group structure such that

- (i) The map $\Phi: (g, m) \mapsto gm$ is smooth.
- (ii) If $X \in \mathfrak{X}(M)$ is complete and tangent to G then the group homomorphism $\varepsilon \mapsto \Psi_\varepsilon$ is a one-parameter subgroup of G .

The above is the desired property entailing uniqueness:

1.3.4 Proposition. *A group of transformations of a T_2 -manifold M admits at most one structure as a Lie group of transformations of M .*

Proof. Let G and G' be two Lie group structures on a group of transformations that both satisfy 1.3.3. We have to show that $\text{id} : G \rightarrow G'$ is a diffeomorphism. We first note that by 1.3.3 (i), G is a Lie transformation group on M , denoted by $\Phi : G \times M \rightarrow M$. Hence by 1.1.13 any $v \in T_e G$ defines a complete vector field $\Phi(v) \in \mathfrak{X}(M)$ with $\text{Fl}_\varepsilon^{\Phi(v)}(m) = \exp(\varepsilon v)m$. It follows that $\Phi(v)$ is tangent to G . Now 1.3.3 (ii) implies that $\varepsilon \mapsto \text{id} \circ \exp(\varepsilon v) : \mathbb{R} \rightarrow G'$ is a one-parameter subgroup of G' . By [11, 8.3] there is a unique $v' \in T_e G'$ such that $\text{id} \circ \exp(\varepsilon v) = \exp'(\varepsilon v')$ (with \exp' the exponential map of G'). Thereby we obtain a well-defined map $v \mapsto v'$ from $T_e G$ to $T_e G'$.

Fixing a basis (v_1, \dots, v_n) of $T_e G$, let $\varphi = (x^1, \dots, x^n)$ be the corresponding canonical chart of the second kind, cf. [11, Rem. 8.5]. Then since id is a group homomorphism,

$$\text{id} \circ (\exp(x^1 v_1) \dots \exp(x^n v_n)) = \exp'(x^1 v'_1) \dots \exp(x^n v'_n),$$

implying that id is smooth on the domain of φ . Smoothness at an arbitrary point of G then follows by writing $\text{id} = L'_g \circ \text{id} \circ L_{g^{-1}}$. Interchanging G and G' in the above argument shows that id in fact is a diffeomorphism. \square

1.3.5 Remark. By 1.3.3 (i), any Lie group G of transformations is also a Lie transformation group that acts effectively on M . Therefore, 1.1.17 implies that $\mathcal{R}(G, M)$ is Lie algebra-isomorphic to \mathfrak{g}_R .

1.3.6 Proposition. *If a group G of transformations of a T_2 -manifold M admits the structure of a Lie group of transformations then the set $\mathfrak{X}_t(M)$ of complete vector fields on M tangent to G is a finite-dimensional Lie algebra (namely $\mathcal{R}(G, M)$).*

Proof. Denote the action of G on M by $\Phi : G \times M \rightarrow M$. By 1.1.13, every $\Phi(v) \in \mathcal{R}(G, M)$ is complete with corresponding transformations $\Psi_\varepsilon : m \mapsto \text{Fl}_\varepsilon^{\Phi(v)}(m) = \exp(\varepsilon v)m$, hence belonging to G . Thus $\Phi(v)$ is a complete vector field on M that is tangent to G , i.e., $\mathcal{R}(G, M) \subseteq \mathfrak{X}_t(M)$.

Conversely, let $X \in \mathfrak{X}_t(M)$ and let $\Psi_\varepsilon := m \mapsto \text{Fl}_\varepsilon^X(m)$ be the corresponding transformation of M . By 1.3.3 (ii), $\varepsilon \mapsto \Psi_\varepsilon$ is a one-parameter subgroup of G , so by [11, 8.3] there is a unique $v \in T_e G$ with $\Psi_\varepsilon(m) = \exp(\varepsilon v)m$ for all $m \in M$. Thus the vector fields X and $\Phi(v)$ have the same maximal integral curves and hence coincide, so $X \in \mathcal{R}(G, M)$. \square

1.3.7 Remark. There is in fact also a converse to 1.3.6, see [15, p. 103]: Let G be a group of transformations of a T_2 -manifold M . If $\mathfrak{X}_t(M)$ generates a finite-dimensional Lie algebra \mathcal{A} of vector fields on M then G admits the structure of a Lie group of transformations of M and $\mathcal{A} = \mathcal{R}(G, M)$.

Chapter 2

Integrability of distributions of non-constant rank

2.1 Distributions of non-constant rank

In [11, 17.32] we proved the classical Frobenius theorem on the integrability of distributions of constant rank:

2.1.1 Theorem. *Let M be an n -dimensional T_2 -manifold and let Δ be a k -dimensional distribution on M . Then the following are equivalent:*

- (i) Δ is involutive.
- (ii) Every point in M lies in the domain of a flat chart $\varphi = (x^1, \dots, x^n)$ for Δ , i.e., such that $\partial_{x^1}|_m, \dots, \partial_{x^k}|_m$ forms a basis of $\Delta(m)$ for each m in the domain of φ .
- (iii) Every point of M is contained in an integral manifold of Δ .
- (iv) Every point m of M lies in a cubic chart $(\varphi = (x^1, \dots, x^n), U)$, $\varphi(U) = [-c, c]^n$ centered around m such that the slices $U_a = \varphi^{-1}(\mathbb{R}^k \times \{a\})$ are integral manifolds of Δ . If M' is a connected integral-manifold of Δ with $M' \subseteq U$ then M' is contained in one such slice.

Note that in [11] we called a distribution integrable if it satisfies (ii). However, in this chapter we will use (iii) instead, cf. 2.1.17 below.

For the application to orbits of (local) transformation groups we have in mind, 2.1.1 is too restrictive in that it requires Δ to have constant rank (i.e., dimension) k everywhere. In this chapter we closely follow H. Sussmann's article [18], as well as P. Michor's exposition in [13] to develop a theory of integrability for distributions of non-constant rank.

Throughout this chapter, all manifolds are supposed to be T_2 and paracompact.

2.1.2 Definition. *We call*

$$\mathfrak{X}_{\text{loc}}(M) := \bigcup \{ \mathfrak{X}(U) \mid U \subseteq M \text{ open} \}$$

the space of local vector fields on M .

If X, Y are local vector fields on M , then so is $[X, Y]$, defined on the intersection of the domains of X and Y . We agree to consider the ‘empty vector field’ an element of $\mathfrak{X}_{\text{loc}}(M)$, to avoid having to add formulations like ‘if the domains of X and Y intersect’, and similar for further local notions to be introduced below.

For any $X \in \mathfrak{X}_{\text{loc}}(M)$ we denote the maximal domain of Fl^X in $\mathbb{R} \times M$ by U_X . U_X is an open subset of $\mathbb{R} \times M$ (see [9, 2.5.17]). For each $t \in \mathbb{R}$, Fl_t^X is a diffeomorphism of some maximal open set $U_t(X)$ (which may be empty) onto some open set $U'_t(X)$. Note that $U'_t(X) = U_{-t}(X)$.

For any $n \geq 1$ in \mathbb{N} , any $\xi = (X_1, \dots, X_n) \in \mathfrak{X}_{\text{loc}}(M)^n$, any $T = (t_1, \dots, t_n) \in \mathbb{R}^n$, and any $m \in M$ we set

$$\xi_T(m) := \text{Fl}_{t_1}^{X_1}(\text{Fl}_{t_2}^{X_2}(\dots \text{Fl}_{t_n}^{X_n}(m) \dots)). \quad (2.1.1)$$

The maximal domain of $(T, m) \mapsto \xi_T(m)$ then is an open subset of $\mathbb{R}^n \times M$ that will be denoted by U_ξ , and we let $U_T(\xi)$ be the set of all $m \in M$ such that $\xi_T(m)$ is defined, i.e., $U_T(\xi) = \{m \in M \mid (T, m) \in U_\xi\}$.

2.1.3 Definition. A diffeomorphism $f: U \rightarrow U'$ between open subsets of M is called a *local diffeomorphism*.

If $f_i: U_i \rightarrow U'_i$ ($i = 1, 2$) are local diffeomorphisms, then so is $f_1 \circ f_2$, with domain $f_2^{-1}(U_1)$ and range $f_1(U'_2 \cap U_1)$. Moreover, $f_1^{-1}: U'_1 \rightarrow U_1$ is a local diffeomorphism as well. A group of local diffeomorphisms is a set G of local diffeomorphisms that is closed under composition and inverses.

Our main examples of local diffeomorphisms are flows of local vector fields. For any $X \in \mathfrak{X}_{\text{loc}}(M)$ the set of all Fl_t^X ($t \in \mathbb{R}$) is called the group of local diffeomorphisms generated by X , and is denoted by G_X . More generally, if D is any subset of $\mathfrak{X}_{\text{loc}}(M)$ then there exists a smallest group of local diffeomorphisms containing $\bigcup\{G_X \mid X \in D\}$. This group (more precisely, pseudogroup) will be denoted by G_D . It is called the group of local diffeomorphisms generated by D . By definition, we have

$$G_D = \{\xi_T \mid \exists n: \xi \in D^n, T \in \mathbb{R}^n\}. \quad (2.1.2)$$

Given finite tuples $\lambda = (\lambda_1, \dots, \lambda_m)$, $\mu = (\mu_1, \dots, \mu_k)$, by $\lambda\mu$ we denote their concatenation $(\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_k)$, and by $\hat{\lambda}$ we denote the reverse tuple $(\lambda_m, \dots, \lambda_1)$. With these notations we have

$$\xi_T \eta_{T'} := \xi_T \circ \eta_{T'} = (\xi\eta)_{TT'} \quad \text{and} \quad (\xi_T)^{-1} = \xi_{-\hat{T}}.$$

A subset D of $\mathfrak{X}_{\text{loc}}(M)$ is called *everywhere defined* if the union of all domains of elements of D covers M . An analogous definition applies to groups of local diffeomorphisms.

2.1.4 Definition. Let G be an everywhere defined group of local diffeomorphisms. Two elements m_1, m_2 of M are called *G-equivalent* if there exists some $g \in G$ such that $g(m_1) = m_2$. The equivalence classes of the resulting equivalence relation on M are called the *orbits* of G . If $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ is everywhere defined then the G_D -orbits are also called *D-orbits*. The orbit of $m \in M$ is called *trivial* if it equals $\{m\}$.

Thus m_1, m_2 belong to the same D -orbit if and only if there exists some $n \geq 1$, some $\xi \in D^n$ and some $T \in \mathbb{R}^n$ such that $\xi_T(m_1) = m_2$. From this we immediately conclude:

2.1.5 Lemma. Two points m_1, m_2 belong to the same D -orbit if and only if there exists a piecewise smooth curve $\gamma: [a, b] \rightarrow M$ such that $\gamma(a) = m_1$, $\gamma(b) = m_2$, with the following property: There exist $a = t_0 < t_1 < \dots < t_r = b$ and $X_i \in D$ ($i = 1, \dots, r$) such that, for each i , $\gamma|_{[t_{i-1}, t_i]}$ is an integral curve of X_i or of $-X_i$.

Such a curve γ will be called a piecewise integral curve of D .

2.1.6 Remark. We now want to endow the orbits of D with a natural topology. To this end, for any $m \in M$ and $\xi \in D^n$, let $\rho_{\xi,m} := T \mapsto \xi_T(m)$, and let $U_{\xi,m} \subseteq \mathbb{R}^n$ be its domain. Then the D -orbit S_m of m is the union of all the images of the mappings $\rho_{\xi,m}$. We equip $S := S_m$ with the finest topology such that each $\rho_{\xi,m}$ ($n \geq 1, \xi \in D^n$) is continuous. Since each $\rho_{\xi,m}$ is continuous for the trace topology of M on S it follows that the topology of S is finer than the trace topology, i.e., $S \hookrightarrow M$ is continuous. In particular, S is T_2 . In general, the topology on S will be strictly finer than the trace topology. Since all $U_{\xi,m}$ are connected and their images all contain m , S is connected.

2.1.7 Remark. The topology on S does not depend on the choice of $m \in S$. To see this, denote by S_m the set S equipped with the topology induced by the maps $\rho_{\xi,m}$. By symmetry, it will suffice to show that $\text{id} : S_m \rightarrow S_{m'}$ is continuous for all $m, m' \in S$. Pick η, T_0 such that $\eta_{T_0}(m') = m$. Then $\rho_{\xi,m} = T \mapsto \xi_T \eta_{T_0}(m')$, which is the composition of $T \rightarrow TT_0$ and $\rho_{\xi\eta,m'}$. Now $\rho_{\xi\eta,m'}$ is continuous into $S_{m'}$, so $\rho_{\xi,m} : U_{\xi,m} \rightarrow S_{m'}$ is continuous. By the universal property of the finest topology, the claim follows.

We next generalize the definition ([11, Def. 17.1]) of distribution to the variable rank setting:

2.1.8 Definition. A distribution on a manifold M is a mapping Δ that assigns to every $m \in M$ a linear subspace $\Delta(m)$ of $T_m M$. A set D of local vector fields is said to span Δ if, for every $m \in M$, $\Delta(m) = \text{span}\{X(m) \mid X \in D\}$.

If $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ is everywhere defined then it spans a unique distribution, which will be denoted by Δ_D . Any distribution which is of the form Δ_D for some everywhere defined family $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ is called smooth.

We say that a local vector field X on M belongs to a distribution Δ if $X(m) \in \Delta(m)$ for every m in the domain of X . Let

$$D_\Delta := \{X \in \mathfrak{X}_{\text{loc}}(M) \mid X \text{ belongs to } \Delta\}.$$

Then Δ is C^∞ if and only if Δ is spanned by D_Δ .

A distribution Δ is called invariant under a group of local diffeomorphisms G if

$$\forall m \in M : \quad T_m g(\Delta(m)) \subseteq \Delta(g(m)) \quad (2.1.3)$$

for all $g \in G$ such that m is in the domain of g . In this case also $T_{gm} g^{-1}$ maps $\Delta(g(m))$ into $\Delta(m)$, i.e., $T_m g(\Delta(m)) = \Delta(g(m))$. It follows that the dimension of $\Delta(m)$ is the same for all points m in the same G -orbit.

A distribution Δ_1 is said to be contained in a distribution Δ_2 if $\Delta_1(m) \subseteq \Delta_2(m)$ for all $m \in M$. If Δ is a distribution and G is a group of local diffeomorphisms on M then there is a smallest distribution Δ^G which contains Δ and is G -invariant (namely the intersection of all such distributions). More precisely, $\Delta^G(m)$ is the linear span of all vectors $v \in T_m M$ such that $v \in \Delta(m)$ or $v = T_{m'} g(w)$ for some $g \in G$ and $m' \in M$ with $w \in \Delta(m')$ and $m = gm'$.

2.1.9 Remark. Let Δ be spanned by $D \subseteq \mathfrak{X}_{\text{loc}}(M)$. Then Δ^G is spanned by the family

$$D \cup \{g_* X \mid X \in D, g \in G \text{ s.t. } g_* X \text{ is defined}\}.$$

It follows that if Δ is C^∞ then so is Δ^G .

Note that if $G = G_D$ then in fact Δ^{G_D} is spanned by

$$\mathcal{D} := \{g_*X \mid X \in D, g \in G \text{ s.t. } g_*X \text{ is defined}\} \quad (2.1.4)$$

(i.e., the union with D is not required). To see this, note that any $X \in D$ with domain, say, U can be written as $X = g_*X$ for $g = \text{Fl}_0^X = \text{id}_U \in G_D$. If $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ then a G_D -invariant distribution is called D -invariant. The smallest D -invariant distribution which contains Δ is denoted by Δ^D , i.e.,

$$\Delta^D := \Delta^{G_D}.$$

Next, let D be an everywhere defined subset of $\mathfrak{X}_{\text{loc}}(M)$. Then we set

$$P_D := \Delta_D^D := (\Delta_D)^D. \quad (2.1.5)$$

Thus P_D is the smallest distribution that is D -invariant and contains Δ_D . By 2.1.9, P_D is smooth and the dimension of $P_D(m)$ depends only on the D -orbit of m .

2.1.10 Remark. It is important to note that the D -orbits are precisely the P_D -orbits: to see this, by (2.1.2) and the remark following 2.1.4 it suffices to note that if $g = \xi_T \in G_D$ and $X \in D$, then $X \sim_{\xi_T} g_*X$, and therefore (by [11, 17.8]),

$$\text{Fl}_t^{g_*X} = \xi_T \circ \text{Fl}_t^X \circ \xi_{-T}.$$

Hence the flows of the g_*X do not alter the D -orbits.

Thus the following definition makes sense:

2.1.11 Definition. Let S be an orbit of an everywhere defined subset D of $\mathfrak{X}_{\text{loc}}(M)$. For any $m \in S$, the dimension of $P_D(m)$ is called the rank of S .

Trivial orbits are characterized by the following result:

2.1.12 Lemma. Let $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ be everywhere defined and let S be the orbit of $m \in M$. The following are equivalent:

- (i) $P_D(m) = \{0\}$.
- (ii) The orbit of m is trivial, i.e., $S = \{m\}$.

Proof. (i) \Rightarrow (ii): By (2.1.4), for any $X \in D$ and $g \in G_D$ such that g_*X is defined, $g_*X(m) = 0$. Therefore, the flow of any element of P_D leaves m unchanged.

(ii) \Rightarrow (i): Suppose that $P_D(m) \neq \{0\}$, then by (2.1.4) there would exist some $g \in G_D$ and some $X \in D$ with $g_*X(m) \neq 0$. But then the flow of g_*X would leave $\{m\}$, contradicting (ii). \square

2.1.13 Example. As in 1.2.7, let $M = \mathbb{R}^2$ and $X := -y\partial_x + x\partial_y$. Now set $D := \{X\}$. Then the D -orbit of any (x_0, y_0) is a circle through (x_0, y_0) with center $(0, 0)$, while the orbit at $(0, 0)$ is trivial. In this example, $P_D = \Delta_D$: In fact, this is always true when D consists of only one vector field X because in that case

$$\text{Fl}_s^{(\text{Fl}_t^X)_*X} = \text{Fl}_t^X \circ \text{Fl}_s^X \circ \text{Fl}_{-t}^X = \text{Fl}_s^X,$$

so X and $(\text{Fl}_t^X)_*X$ have the same flow and therefore coincide.

2.1.14 Definition. A set $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ is called involutive if for any $X, Y \in \mathfrak{X}_{\text{loc}}(M)$ that belong to D also $[X, Y]$ belongs to D . If D is any subset of $\mathfrak{X}_{\text{loc}}(M)$ then the smallest involutive subset of $\mathfrak{X}_{\text{loc}}(M)$ that contains D will be denoted by D^* . A smooth distribution Δ is called involutive if the corresponding set D_Δ is involutive.

2.1.15 Lemma. Let D be an everywhere defined subset of $\mathfrak{X}_{\text{loc}}(M)$. Then

$$\Delta_D \subseteq \Delta_{D^*} \subseteq P_D.$$

Proof. The first inclusion is immediate since $D \subseteq D^*$. The second inclusion follows since P_D is involutive by 2.2.4 (iv) below. \square

2.1.16 Example. Clearly the first inclusion in 2.1.15 can be proper. This example shows that the same may happen for the second inclusion. Let $M = \mathbb{R}^2$, $X_1 := \frac{\partial}{\partial x}$, $X_2 = \phi \frac{\partial}{\partial y}$, where $\phi(x, y) = \psi(x)$ and ψ is a smooth function with $\psi(x) = 0$ for $x \leq 0$ and $\psi(x) > 0$ for $x > 0$. Let $D := \{X_1, X_2\}$. Then since $D(x, y)$ has dimension 2 for $x > 0$, the same is true for $P_D \supseteq \Delta_D$. Moreover, any point in \mathbb{R}^2 can be joined to a point (x, y) with $x > 0$ by a piecewise integral curve of D , so in fact P_D has dimension 2 everywhere. However, for $x \leq 0$ the distribution Δ_{D^*} has dimension 1.

2.1.17 Definition. An immersive submanifold S of M is called an integral manifold of a distribution Δ on M if, for all $s \in S$, we have $T_s S = \Delta(s)$. A C^∞ -distribution Δ is said to be integrable, or to have the integral manifold property if, for every $m \in M$, either the orbit of m is trivial or there exists an integral manifold S of Δ such that $m \in S$.

Note that we notationally suppress the inclusion map $j : S \hookrightarrow M$ here. If a C^∞ -distribution Δ is integrable then a smooth vector field X belongs to Δ if and only if X is tangent to every integral manifold of Δ . By [11, 17.16], any such X can locally be viewed as a smooth vector field on any given integral manifold. Moreover, by [11, 17.22] and 2.1.12 we have:

2.1.18 Lemma. Any integrable C^∞ -distribution is involutive.

Note, however, that the converse is not true in the present situation (contrary to constant rank distributions!). In fact, 2.1.16 provides an example of a distribution Δ_D that is involutive:

$$[X_1, X_2](x, y) = \psi'(x) \partial_y = \begin{cases} \frac{\psi'(x)}{\psi(x)} X_2 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

but cannot have the integral manifold property (the dimension of the integral manifolds at $x = 0$ would have to be 1 and 2).

2.1.19 Definition. Let Δ be a smooth distribution on M . A maximal integral manifold of Δ is a connected immersive submanifold S of M such that

- (i) S is an integral manifold of Δ , and
- (ii) every connected integral submanifold of Δ that intersects S is an open submanifold of S .

Δ is said to have the maximal integral manifold property if through each point of M with $P_{D\Delta}(m) \neq 0$ (cf. 2.1.12) there passes a maximal integral manifold of Δ .

In particular, any two maximal integral submanifolds through the same point m must coincide.

For the discussion below we will need the following auxiliary result:

2.1.20 Lemma. *Let X, Y be smooth local vector fields on M . Then for any m in the intersection of the domains of X, Y we have:*

$$[X, Y](m) = \frac{d}{dt} \Big|_0 \text{Fl}_{-\sqrt{t}}^Y(\text{Fl}_{-\sqrt{t}}^X(\text{Fl}_{\sqrt{t}}^Y(\text{Fl}_{\sqrt{t}}^X(m))))$$

Proof. For any local smooth function f we have

$$\frac{d}{dt}(\text{Fl}_t^X)^* f = X(f) \circ \text{Fl}_t^X = (\text{Fl}_t^X)^*(L_X f) = (\text{Fl}_t^X)^*(Tf(X)). \quad (2.1.6)$$

Since

$$\frac{d}{dt} \Big|_0 f(\text{Fl}_{-\sqrt{t}}^Y(\text{Fl}_{-\sqrt{t}}^X(\text{Fl}_{\sqrt{t}}^Y(\text{Fl}_{\sqrt{t}}^X(m)))) = Tf \left(\frac{d}{dt} \Big|_0 \text{Fl}_{-\sqrt{t}}^Y(\text{Fl}_{-\sqrt{t}}^X(\text{Fl}_{\sqrt{t}}^Y(\text{Fl}_{\sqrt{t}}^X(m)))) \right)$$

and $Tf([X, Y])(m) = ([X, Y](f))(m)$, the claim will follow if we can show that for any such f we have

$$\frac{d}{dt} \Big|_0 \left((\text{Fl}_{\sqrt{t}}^X)^*(\text{Fl}_{\sqrt{t}}^Y)^*(\text{Fl}_{-\sqrt{t}}^X)^*(\text{Fl}_{-\sqrt{t}}^Y)^* f \right) (m) = ([X, Y](f))(m).$$

Using (2.1.6) we obtain

$$\begin{aligned} \frac{d}{dt} & \left((\text{Fl}_{\sqrt{t}}^X)^*(\text{Fl}_{\sqrt{t}}^Y)^*(\text{Fl}_{-\sqrt{t}}^X)^*(\text{Fl}_{-\sqrt{t}}^Y)^* f \right) \\ &= ((\text{Fl}_{\sqrt{t}}^X)^* L_X ((\text{Fl}_{\sqrt{t}}^Y)^*(\text{Fl}_{-\sqrt{t}}^X)^*(\text{Fl}_{-\sqrt{t}}^Y)^* f) \\ &\quad + (\text{Fl}_{\sqrt{t}}^X)^*(\text{Fl}_{\sqrt{t}}^Y)^* L_Y ((\text{Fl}_{-\sqrt{t}}^X)^*(\text{Fl}_{-\sqrt{t}}^Y)^* f) \\ &\quad - (\text{Fl}_{\sqrt{t}}^X)^*(\text{Fl}_{\sqrt{t}}^Y)^*(\text{Fl}_{-\sqrt{t}}^X)^* L_X ((\text{Fl}_{-\sqrt{t}}^Y)^* f) \\ &\quad - (\text{Fl}_{\sqrt{t}}^X)^*(\text{Fl}_{\sqrt{t}}^Y)^*(\text{Fl}_{-\sqrt{t}}^X)^*(\text{Fl}_{-\sqrt{t}}^Y)^*(L_Y f)) \cdot \frac{1}{2\sqrt{t}} =: \frac{g(\sqrt{t})}{2\sqrt{t}} \end{aligned}$$

We need to calculate the limit as $t \searrow 0$ of this expression. Now since $g(0) = 0$ it follows that

$$\lim_{t \searrow 0} \frac{g(\sqrt{t})}{2\sqrt{t}} = \frac{1}{2} g'(0).$$

Again using (2.1.6) we calculate:

$$\begin{aligned} g'(0) &= (L_X L_X + L_X(L_Y - L_X - L_Y) + (L_X + L_Y)L_Y + L_Y(-L_X - L_Y) \\ &\quad - (L_X + L_Y - L_X)L_X + L_X L_Y - L_X L_Y - L_Y L_Y + L_X L_Y + L_Y L_Y) f \\ &= 2(L_X L_Y - L_Y L_X) f = 2[X, Y]f, \end{aligned}$$

giving the claim. \square

2.1.21 Remark. To clarify the geometric meaning of Δ_{D^*} and P_D , and also to motivate the structure of the main results in the following section, suppose that a subset D of $\mathfrak{X}_{\text{loc}}(M)$ is given and that we want to find a distribution Δ with the property that the orbits of D are precisely the maximal integral manifolds of Δ . It is then geometrically natural to define $\Delta(m)$ as the set of all tangent vectors of smooth curves that pass through m and lie entirely in the D -orbit of m . Call this set of curves Γ_m .

For any $X \in D$, the integral curve $t \mapsto \text{Fl}_t^X(m)$ belongs to Γ_m . Consequently, $\Delta(m)$ must contain $\frac{d}{dt}\big|_0 \text{Fl}_t^X(m) = X(m)$. Moreover, for $X, Y \in D$ the curve

$$t \mapsto \text{Fl}_{-\sqrt{t}}^Y(\text{Fl}_{-\sqrt{t}}^X(\text{Fl}_{\sqrt{t}}^Y(\text{Fl}_{\sqrt{t}}^X(m))))$$

belongs to Γ_m . By 2.1.20, the derivative of this curve at $t = 0$ is $[X, Y](m)$, which therefore must also lie in $\Delta(m)$. Iterating this procedure it follows that Δ_{D^*} must be contained in Δ .

There may, however, be further vectors beside those in $\Delta_{D^*}(m)$ that have to be contained in $\Delta(m)$: Let $X \in D$, fix $t' \in \mathbb{R}$ and set $m' := \text{Fl}_{-t'}^X(m)$. If γ is a smooth curve with $\gamma(0) = m'$ and $\gamma \in \Gamma_{m'}$ then the curve $\sigma := t \mapsto \text{Fl}_{t'}^X(\gamma(t))$ belongs to Γ_m . Setting $v := \gamma'(0)$ it follows that $\sigma'(0) = T_{m'}\text{Fl}_{t'}^X(v)$. We conclude that for any $v \in \Delta(m')$ we must have $T_{m'}\text{Fl}_{t'}^X(v) \in \Delta(m)$, i.e., Δ must be D -invariant (i.e., G_D -invariant).

These considerations suggest to define Δ as the smallest D -invariant distribution that contains Δ_{D^*} . We shall see below that this distribution coincides with the smallest D -invariant distribution that contains Δ_D , i.e., with P_D .

The above also explains why Δ_{D^*} by itself may be too small to serve our purpose: it may not contain sufficiently many directions: one may move within the orbit of m along an integral curve of some $X \in D$, catch up a new direction there and come back. Only P_D is large enough to also contain these directions.

2.2 Orbit structure and integrability

Throughout this section, let D be an everywhere defined subset of $\mathfrak{X}_{\text{loc}}(M)$, and let S be an orbit of D (cf. 2.1.12). We equip S with the natural topology introduced in 2.1.6, and we will use the notations introduced there. We set

$$D^\infty := \bigcup_{n \in \mathbb{N}_{>0}} D^n,$$

the set of all finite tuples of elements of D . If $\xi \in D^n$ then $U_{\xi, m}$ is open in \mathbb{R}^n and $\rho_{\xi, m} : U_{\xi, m} \rightarrow M$, $T \mapsto \xi_T(m)$ is \mathcal{C}^∞ . Also, for any $m \in S$, $\rho_{\xi, m} : U_{\xi, m} \rightarrow S$ is continuous.

Given $\xi \in D^\infty$, $m \in M$, and $T \in U_{\xi, m}$, we set

$$V(\xi, m, T) := T_T \rho_{\xi, m}(T_T U_{\xi, m}),$$

the image of the tangent space of $U_{\xi, m}$ at T under the tangent map of $\rho_{\xi, m}$. Setting $m_0 := \xi_T(m)$, $V(\xi, m, T)$ is a linear subspace of $T_{m_0}M$.

2.2.1 Lemma. *Let $\xi \in D^\infty$, $m \in S$, $T \in U_{\xi, m}$, and $m_0 := \xi_T(m)$. Then*

$$V(\xi, m, T) \subseteq P_D(m_0).$$

Proof. To begin with, let $n = 1$, $\xi = X$ and $T = t_0$. Then $U_{\xi,m} = \{t \in \mathbb{R} \mid \exists \text{Fl}_t^X(m)\}$ and $T_{t_0}U_{\xi,m} = \mathbb{R}$. Also, $\rho_{\xi,m}(t) = \text{Fl}_t^X(m)$, so setting $m' := \text{Fl}_{t_0}^X(m)$ we have $T_{t_0}\rho_{\xi,m} = X(m')$ and therefore

$$V(X, m, t_0) = \text{span}(X(m')) \subseteq \Delta_D(m') \subseteq P_D(m').$$

Suppose now that the claim is already true for $n - 1$. Let $\xi \in D^n$ and $T \in U_{\xi,m}$. Then we may write $\xi = X\eta$, $T = t_0T'$ for suitable $X \in D$ and $\eta \in D^{n-1}$, $T' \in U_{\eta,m}$ and $t_0 \in \mathbb{R}$. By definition, $V(\xi, m, T) = \text{im}(T_T\rho_{\xi,m})$. Here, we have $\rho_{\xi,m}(t_0, T') = \text{Fl}_{t_0}^X(\rho_{\eta,m}(T'))$, so that

$$\begin{aligned} T_{(t_0, T')}\rho_{\xi,m} &= X(\text{Fl}_{t_0}^X(\rho_{\eta,m}(T'))) \oplus T_{\rho_{\eta,m}(T')}\text{Fl}_{t_0}^X(T_{T'}\rho_{\eta,m}) \\ &= X(\xi_T(m)) \oplus T_{\rho_{\eta,m}(T')}\text{Fl}_{t_0}^X(T_{T'}\rho_{\eta,m}). \end{aligned} \quad (2.2.1)$$

Therefore, $V(\xi, m, T) \subseteq \text{span}(X(\xi_T(m))) + T_{\rho_{\eta,m}(T')}\text{Fl}_{t_0}^X(V(\eta, m, T'))$. By our induction assumption, $V(\eta, m, T') \subseteq P_D(\eta_{T'}(m))$, and since P_D is D -invariant we obtain

$$T_{\rho_{\eta,m}(T')}\text{Fl}_{t_0}^X(V(\eta, m, T')) \subseteq P_D(\xi_T(m)).$$

Moreover, $X(\xi_T(m)) \in \Delta_D(\xi_T(m)) \subseteq P_D(\xi_T(m))$, so altogether $V(\xi, m, T) \subseteq P_D(\xi_T(m))$, concluding the proof for n . \square

2.2.2 Lemma. *Let $m_0 \in S$. Then there exist $\xi \in D^\infty$, $m \in S$, and $T \in U_{\xi,m}$ such that $\xi_T(m) = m_0$ and $V(\xi, m, T) = P_D(m_0)$.*

Proof. We will see that the claim follows once we establish the following two statements:

- (i) If $\xi, \eta \in D^\infty$, $m, m' \in S$, $T \in U_{\xi,m}$, and $T' \in U_{\eta,m'}$ are such that $\xi_T(m) = \eta_{T'}(m')$ then there exist $\sigma \in D^\infty$, $m'' \in S$, and $T'' \in U_{\sigma,m''}$ such that

$$V(\xi, m, T) \cup V(\eta, m', T') \subseteq V(\sigma, m'', T'').$$

- (ii) There exists a finite subset A of $P_D(m_0)$ that spans $P_D(m_0)$ and satisfies: for every $v \in A$ there exist $\xi \in D^\infty$, $m \in S$, and $T \in U_{\xi,m}$ such that $\xi_T(m) = m_0$ and $v \in V(\xi, m, T)$.

In fact, let us suppose that (i) and (ii) have already been established. By (ii), any element of $P_D(m_0)$ is a linear combination of the elements of $A = \{v_1, \dots, v_k\}$ and there exist ξ_i, m_i, T_i ($1 \leq i \leq k$) such that $v_i \in V(\xi_i, m_i, T_i)$ and $\xi_{T_i}(m_i) = m_0$ for all i . By (i), then, there exists one $V(\xi, m, T)$ containing all $V(\xi_i, m_i, T_i)$, so $P_D(m_0) \subseteq V(\xi, m, T)$. Together with 2.2.1 this finishes the proof. It therefore only remains to show (i) and (ii).

To see (i), set $m'' := m'$, $\sigma := \xi\hat{\xi}\eta$, and $T'' := T(-\hat{T})T'$. Then

$$\rho_{\sigma,m''}(T'') = \sigma_{T''}(m'') = \xi_T(\hat{\xi}_{-\hat{T}}(\eta_{T'}(m'))) = \eta_{T'}(m') = m_0.$$

Analogously to (2.2.1) we may therefore split the tangent map of $\rho_{\sigma,m''}$ at T'' into a direct sum of three maps, corresponding to differentiation with respect to T , $-\hat{T}$, and T' , respectively. Therefore $V(\sigma, m'', T'') = \text{im}(T_{T''}\rho_{\sigma,m''})$ contains the sum of the images of these partial maps. Since $\hat{\xi}_{-\hat{T}}\eta_{T'}(m'') = m$, the first of these maps is $T_T\rho_{\xi,m}$, so $V(\xi, m, T) = \text{im}(T_T\rho_{\xi,m}) \subseteq V(\sigma, m'', T'')$. Moreover, since $\xi_T\hat{\xi}_{-T}$ is

the identity, the third of these maps has image $\text{im}(T_{T'}\rho_{\eta,m'}) = V(\eta, m', T')$, so also $V(\eta, m', T') \subseteq V(\sigma, m'', T'')$. This proves (i).

To see (ii), let \tilde{A} be the set of all vectors $Y(m_0)$, where Y is of the form g_*X for some $X \in D$ and some $g \in G_D$. By (2.1.4), \tilde{A} spans $P_D(m_0)$. Since $\dim P_D(m_0) \leq \dim(M)$ we may extract a finite subset A from \tilde{A} that still spans $P_D(m_0)$. Given any $v \in A$ we have $v = T_m g(w)$, where $g \in G_D$, $m \in S$, $g(m) = m_0$ and $w \in T_m M$ is of the form $w = X(m)$ for some $X \in D$. Also, since $g \in G_D$ we have $g = \xi_T$ for some $\xi \in D^\infty$ and some $T \in U_{\xi,m}$.

Now set $\eta := \xi X$ and $T' := (T, 0)$. Then $\eta_{T'}(m) = \xi_T(m) = m_0$, so $\rho_{\eta,m}(T') = \eta_{T'}(m) = \xi_T \circ \text{Fl}_0^X(m)$, and splitting the tangent map as above we find

$$T_{T'}\rho_{\eta,m} = T_T\rho_{\xi,m} \oplus T_m\xi_T(X(m)).$$

In particular, $v = T_m\xi_T(X(m)) \in \text{im}(T_{T'}\rho_{\eta,m}) = V(\eta, m, T')$, which gives (ii). \square

2.2.3 Lemma. *If a connected integral manifold N of P_D intersects S then N is an open subset of S (in the topology introduced in 2.1.6).*

Proof. Note first that if N intersects S in, say, m then S must be non-trivial by 2.1.12 because $1 \leq \dim T_m N$ and $T_m N = P_D(m)$. Let \mathcal{D} as in (2.1.4), then from 2.1.10 we know that the \mathcal{D} -orbits are precisely the D -orbits.

By (2.1.4), for any $m \in N$ there exist $X_1, \dots, X_p \in \mathcal{D}$ such that $\{X_1(m), \dots, X_p(m)\}$ form a basis for $P_D(m) = T_m N$. In particular, $p = \dim(N)$. Let

$$\begin{aligned} \Phi: \mathbb{R}^p &\rightarrow N \\ (t_1, \dots, t_p) &\mapsto \text{Fl}_{t_1}^{X_1} \circ \text{Fl}_{t_2}^{X_2} \circ \dots \circ \text{Fl}_{t_p}^{X_p}(m). \end{aligned}$$

Then Φ is a diffeomorphism from an open neighborhood of 0 in \mathbb{R}^p onto a neighborhood of m in N : In fact, since X_1, \dots, X_p are tangential to N it follows from [11, 17.16] that their restrictions to N can be viewed as vector fields on N , and since $\partial_i \Phi(0) = X_i(m)$, the inverse function theorem gives the claim.

From what was said above it follows that every point in the image of Φ lies in the same D -orbit as m . Thus any point in N has a neighborhood that is contained in one orbit of D . Since N is connected, it follows that N is contained in a single D -orbit S (given $n_1, n_2 \in N$ we may connect them by a smooth curve γ in N . Now covering γ by neighborhoods as above it follows that the entire curve lies in the same orbit). Consequently, if $N \cap S \neq \emptyset$ for some D -orbit S , then $N \subseteq S$.

It remains to show that N (as a set) is open in S . By 2.1.6 we need to see that for any $m \in S$ and $\xi \in D^n$ ($n \in \mathbb{N}$), $\rho_{\xi,m}^{-1}(N)$ is open in \mathbb{R}^n . Thus let $T = (t_1, \dots, t_n) \in U_{\xi,m} \subseteq \mathbb{R}^n$ be such that $\rho_{\xi,m}(T) \in N$. Since $\xi = (X_1, \dots, X_n) \in D^n$, it follows that

$$\rho_{\xi,m}(T) = \text{Fl}_{t_1}^{X_1} \circ \text{Fl}_{t_2}^{X_2} \circ \dots \circ \text{Fl}_{t_n}^{X_n}(m).$$

Note that $X_i \in D \subseteq \mathcal{D}$ for all i , so as in our considerations concerning Φ above we may view the restrictions to N of the X_i as local vector fields on N . Thus for $1 \leq i \leq n$, $X_i|_N \in \mathfrak{X}_{\text{loc}}(N)$. As in (2.1.1) it follows from this that the maximal domain of the map $\mathbb{R}^n \times N \rightarrow N$, $(T, m') \mapsto \rho_{\xi,m'}(T)$ is open in $\mathbb{R}^n \times N$, so in particular there exists an open neighborhood of T in \mathbb{R}^n which is mapped by $\rho_{\xi,m}$ into N . We conclude that $\rho_{\xi,m}^{-1}(N)$ is open in \mathbb{R}^n , as claimed. \square

After these preparations we are now ready to prove the first main result of this section:

2.2.4 Theorem. *Let $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ be everywhere defined. Then*

- (i) *If S is a non-trivial orbit of D , then S , equipped with the topology from 2.1.6, admits a unique C^∞ -structure such that S becomes an immersive submanifold of M . The dimension of S equals the rank of S (see 2.1.11).*
- (ii) *With the topology and differentiable structure from (i), each non-trivial orbit of D is a maximal integral submanifold of P_D . In fact, the non-trivial orbits of D are exactly the maximal integral submanifolds of P_D .*
- (iii) *P_D has the maximal integral manifold property.*
- (iv) *P_D is involutive.*

Proof. We note first that (iii) is immediate from (ii), and that (iv) follows from (iii) via 2.1.18.

(i) Let $m_0 \in S$. Then by 2.2.2 there exist $m \in S$, $\xi \in D^\infty$, and $T \in U_{\xi,m}$ such that $\xi_T(m) = m_0$ and $V(\xi, m, T) = P_D(m_0)$. It follows that

$$k := \text{rk}(S) = \dim(P_D(m_0)) = \dim(V(\xi, m, T)) = \dim(\text{im}(T_T \rho_{\xi,m})),$$

so $\text{rk}_T(\rho_{\xi,m}) = k$. As we noted before 2.1.11, $k = \dim(P_D(p))$ for any $p \in S$. Therefore, if $T' \in U_{\xi,m}$ then since $\text{im}(T_{T'} \rho_{\xi,m}) = V(\xi, m, T') \subseteq P_D(\rho_{\xi,m}(T'))$ by 2.2.1, it follows that $\text{rk}(T_{T'} \rho_{\xi,m}) \leq k$, i.e., the rank of $\rho_{\xi,m}$ cannot exceed k anywhere in $U_{\xi,m}$.

On the other hand, the rank of $\rho_{\xi,m}$ locally can only increase, so by the rank theorem ([10, 1.1.3]) there exist open neighborhoods U of T in \mathbb{R}^n and V of m_0 in M and diffeomorphisms $\varphi : U \rightarrow \varphi(U) \subseteq \mathbb{R}^n$ (where n is such that $\xi \in D^n$), $\varphi(T) = 0$ $\psi : V \rightarrow \psi(V) \subseteq \mathbb{R}^l$ (with $l = \dim(M)$), $\psi(m_0) = 0$, such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\rho_{\xi,m}} & V \\ \varphi \downarrow & & \downarrow \psi \\ (-1, 1)^n & \xrightarrow{i_{n,l,k}} & (-1, 1)^l \end{array} \quad (2.2.2)$$

Here, $i_{n,l,k}$ is the map

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k, \underbrace{0, \dots, 0}_{l-k}).$$

Denote by N the (regular) submanifold $\psi^{-1}(i_{n,l,k}((-1, 1)^n))$ of M . Then, as a set, $N = \rho_{\xi,m}(U)$. If $T' \in U$ and $m' := \rho_{\xi,m}(T')$, then since (2.2.2) commutes we obtain

$$T_{m'} N = T \psi^{-1}(i_{n,l,k}(T_{T'} \varphi(\mathbb{R}^n))) = T_{T'} \rho_{\xi,m}(\mathbb{R}^n) = V(\xi, m, T').$$

By 2.2.1, $V(\xi, m, T') \subseteq P_D(m')$ and since both spaces have dimension k they must be equal. Thus N is an integral manifold of P_D . By definition of $\rho_{\xi,m}$, $N \subseteq S$, so 2.2.3 shows that, as a set, N is open in S . In the same way, 2.2.3 applies to any open connected (in the natural manifold topology of N) subset of N . As these sets form a basis of the manifold topology of N , the natural inclusion map $I : N \hookrightarrow S$ is an open map. Moreover, due to (2.2.2) I can be decomposed as

$$\begin{array}{ccccc} U & \xrightarrow{\rho_{\xi,m}} & \rho_{\xi,m}(U) = N & \xrightarrow{I'} & V \\ \varphi \downarrow & & & & \downarrow \psi \\ (-1, 1)^n & \xleftarrow{i_{l,n,k}} & & & (-1, 1)^l \end{array}$$

Here, I' is the inclusion map and (reversing the roles of l and n above)

$$i_{l,n,k} = (x_1, \dots, x_l) \mapsto (x_1, \dots, x_k, \underbrace{0, \dots, 0}_{n-k})$$

(Note that $k \leq \min(l, n)$ by the above). Hence

$$I = \rho_{\xi,m} \circ \varphi^{-1} \circ i_{l,n,k} \circ \psi \circ I'$$

By definition of the topology on S , $\rho_{\xi,m}$ is continuous as a map into S , so also I is. We conclude that I is a homeomorphism onto its image and that this image is open in S .

Denote by \mathcal{N} the family of all manifolds N constructed as above. Also, let $\text{set}(N)$ denote the underlying set of the manifold N . By the above, $\{\text{set}(N) \mid N \in \mathcal{N}\}$ is an open cover of S and each $I : N \hookrightarrow S$ is a homeomorphism onto its image. This provides a family of differentiable structures on the elements of an open cover of S . Our aim is to define a differentiable structure on S such that each $N \in \mathcal{N}$ becomes an open submanifold of S . To this end it suffices to show that for any $N_1, N_2 \in \mathcal{N}$ the differentiable structures of $\text{set}(N_1) \cap \text{set}(N_2)$ as an open submanifold of N_1 resp. N_2 coincide. Call these manifolds W_1, W_2 and let $j : W_1 \rightarrow W_2$ be the identity. By symmetry we only have to show that j is \mathcal{C}^∞ . Since both $i_1 : W_1 \hookrightarrow M$ and $i_2 : W_2 \hookrightarrow M$ are immersions, by [10, 1.1.8] to see this it suffices to show that j is continuous. This, however, is immediate since both i_1 and i_2 are homeomorphisms onto the same open subset of S and $j = i_2^{-1} \circ i_1$.

It follows that S possesses a structure of a smooth manifold whose natural manifold topology is precisely the topology from 2.1.6. Also, since each N as above is an open submanifold of M it follows that S itself is an immersive submanifold of M which is an integral manifold of P_D . Suppose that S' is another such smooth structure on S . Then $i : S \rightarrow S'$ is a homeomorphism and both $S \hookrightarrow M$ and $S' \hookrightarrow M$ are immersions, so again by [10, 1.1.8] i is a diffeomorphism. Hence the smooth structure on S is unique.

(ii) That S is an integral manifold of P_D was already shown in (i). In particular, $\dim(S) = k = \dim(P_D(m_0))$. Also, S is connected by 2.1.6. Now let R be any connected integral manifold of P_D with $R \cap S \neq \emptyset$. By 2.2.3, $\text{set}(R)$ is an open subset of S . As we did in the proof of (i) for S , we may apply 2.2.3 to each open connected subset of R to see that the inclusion map $R \hookrightarrow S$ is an open map. Denote by R' the open submanifold of S with underlying set $\text{set}(R)$ and let $I : R' \rightarrow R$ be the identity map. Then by what we have just shown, I is continuous. Note that both $i_R : R \hookrightarrow M$ and $i_{R'} : R' \hookrightarrow M$ are immersions, and $i_{R'} = i_R \circ I$. Therefore, again by [10, 1.1.8] it follows that I is smooth, and is in fact an immersion. Since both R and R' , being integral manifolds of P_D , have the same dimension k , I is even a local diffeomorphism and, due to its injectivity, a diffeomorphism. We conclude that $R = R'$ as a manifold, i.e., R is an open submanifold of S . Thus S is indeed a maximal integral manifold of P_D , as claimed. Finally, let $m \in M$ and let R be a maximal integral manifold of P_D containing m . Then by what we have just shown, R is contained in the D -orbit S of m . Since, conversely, S is a connected integral manifold of P_D and R is maximal, we in fact have $R = S$. Thus the maximal integral manifolds of P_D are precisely the orbits of D . \square

The second main result is as follows:

2.2.5 Theorem. *Let Δ be a smooth distribution on M and let $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ span Δ (in particular, D is everywhere defined). Then the following statements are equivalent:*

- (i) Δ is integrable.
- (ii) Δ has the maximal integral manifold property.
- (iii) Δ is D -invariant.
- (iv) For every $X \in D$, $t \in \mathbb{R}$, and $m \in M$ such that $\text{Fl}_t^X(m)$ is defined,

$$T_m \text{Fl}_t^X(\Delta(m)) \subseteq \Delta(\text{Fl}_t^X(m)).$$

- (v) For every $m \in M$ there exists $k \geq 1$ and elements X^1, \dots, X^k of D such that

- (a) $\Delta(m) = \text{span}(X^1(m), \dots, X^k(m))$, and
- (b) for every $X \in D$ there exists some $\varepsilon > 0$ and smooth functions $f_j^i : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ ($1 \leq i, j \leq k$) such that

$$[X, X^i](\text{Fl}_t^X(m)) = \sum_{j=1}^k f_j^i(t) X^j(\text{Fl}_t^X(m))$$

for $i = 1, \dots, k$ and $t \in (-\varepsilon, \varepsilon)$.

- (vi) $\Delta = P_D$.

Proof. Clearly, (iv) \Rightarrow (iii) \Rightarrow (vi), and (ii) \Rightarrow (i). Also, (vi) \Rightarrow (ii) follows from 2.2.4.

(i) \Rightarrow (v): Let $m \in M$. If $P_D(m) = 0$ then we may pick any $X^1 \in D$. Then $X^1(m) = 0$, and (v) (a) holds trivially. Next, let $X \in D$ and set $f_1^1 \equiv 0$. By (i) also $[X, X^1] \in D$, so $[X, X^1](m) \in \Delta(m) = \{0\}$. Also, $X(m) = 0$, so $\text{Fl}_t^X(m) = m$ for all t and also (v) (b) is satisfied trivially.

So let $P_D(m) \neq 0$ and let S be an integral manifold of Δ through m . Also, let X^1, \dots, X^k be elements of D such that $\{X^1(m), \dots, X^k(m)\}$ is a basis of $\Delta(m) = T_m S$. Then $k = \dim(S)$ and the restriction of the X^i to S can be viewed as elements of $\mathfrak{X}_{\text{loc}}(S)$. By continuity of the determinant it follows that there exists a neighborhood U of m in S such that $\{X^1(m'), \dots, X^k(m')\}$ is a basis of $T_{m'} S = \Delta(m')$ for all $m' \in U$. If $X \in D$ then X is tangent to S , hence also restricts to a local vector field on S . It follows that also all $[X, X^i]$ are tangent to S , and so they must be smooth linear combinations of the X^i on U . This implies (v) once we pick $\varepsilon > 0$ so small that the curve $t \mapsto \text{Fl}_t^X(m)$ remains in U .

(v) \Rightarrow (iv): By the flow property, it suffices to show (iv) for $t \in (-\varepsilon, \varepsilon)$. For any such t , let $W^i(t) := (\text{Fl}_t^X)^*(X^i)(m)$. Then by [11, 17.7] we get

$$\frac{dW^i}{dt} = (\text{Fl}_t^X)^*(L_X X^i)(m) = (\text{Fl}_t^X)^*([X, X^i])(m) = \sum_{j=1}^k f_j^i(t) (\text{Fl}_t^X)^*(X^j)(m).$$

Hence W^1, \dots, W^k satisfy the system of linear ODE

$$\frac{dW^i}{dt} = \sum_{j=1}^k f_j^i(t) W^j(t)$$

on $(-\varepsilon, \varepsilon)$. Since $\{W^1(0), \dots, W^k(0)\}$ forms a basis of $\Delta(m)$, the same is therefore true for $\{W^1(t), \dots, W^k(t)\}$ for any $t \in (-\varepsilon, \varepsilon)$. Now since $X^i \in D$,

$$T_m \text{Fl}_t^X(W^i(t)) = X^i(\text{Fl}_t^X(m)) \in \Delta(\text{Fl}_t^X(m)),$$

so indeed $T_m \text{Fl}_t^X(\Delta(m)) \subseteq \Delta(\text{Fl}_t^X(m))$. \square

Next, we want to connect the above results to the approach taken in [13]. To do this, we need to introduce some notions first.

2.2.6 Definition. $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ is called *stable* if for all $X, Y \in D$ we have $(\text{Fl}_t^X)^*Y \in D$, for all t such that this expression is defined. A local vector field X on M is called an *infinitesimal automorphism* of a distribution Δ if $T_m \text{Fl}_t^X(\Delta(m)) \subseteq \Delta(\text{Fl}_t^X(m))$ whenever defined. The set of infinitesimal automorphisms of Δ is denoted by $\text{aut}(\Delta)$.

For any $D \subseteq \mathfrak{X}_{\text{loc}}(M)$, let $\mathcal{D} \subseteq \mathfrak{X}_{\text{loc}}(M)$ be as in (2.1.4) (for $G = G_D$).

2.2.7 Lemma. Let $D \subseteq \mathfrak{X}_{\text{loc}}(M)$. Then

- (i) \mathcal{D} is the smallest stable subset of $\mathfrak{X}_{\text{loc}}(M)$ containing D .
- (ii) D is stable if and only if $\mathcal{D} = D$.
- (iii) If D is stable then $P_D = \Delta_D$.

Proof. (i) By (2.1.4), $D \subseteq \mathcal{D}$. Moreover, \mathcal{D} is stable: if $g_*X, h_*Y \in \mathcal{D}$ (for $g = \xi_T, h = \eta_{T'} \in G_D$ and $X, Y \in D$) then by 2.1.10 we obtain

$$(\text{Fl}_t^{g_*X})^*(h_*Y) = (\xi_T \circ \text{Fl}_t^X \circ \xi_{-T'})^*(h_*Y) = (\eta_{-T'} \circ \xi_T \circ \text{Fl}_t^X \circ \xi_{-T'})^*Y \in \mathcal{D}.$$

On the other hand, if $D' \supseteq D$ is stable then given $X \in D$ and $g = \xi_T \in G_D$ for $T = (t_1, \dots, t_n)$ then $(\text{Fl}_{-t_n}^{X_n})^*X \in D'$, so $(\text{Fl}_{-t_{n-1}}^{X_{n-1}})^*(\text{Fl}_{-t_n}^{X_n})^*X \in D'$, etc., leading to $g_*X = (\text{Fl}_{-t_1}^{X_1})^* \dots (\text{Fl}_{-t_n}^{X_n})^*X \in D'$, i.e., $\mathcal{D} \subseteq D'$.

(ii) is immediate from (i), and (iii) follows since $\Delta_{\mathcal{D}} = P_D$ by (2.1.4). \square

After these preparations, we have (see [13, 3.24]):

2.2.8 Theorem. Let M be a smooth manifold and let Δ be a smooth distribution on M . Then the following statements are equivalent:

- (i) Δ is integrable.
- (ii) D_{Δ} is stable.
- (iii) There exists a subset $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ such that \mathcal{D} spans Δ (i.e., s.t. $P_D = \Delta$).
- (iv) $\text{aut}(\Delta) \cap D_{\Delta}$ spans Δ .

Proof. (i) \Leftrightarrow (iii): this is 2.2.5, (i) \Leftrightarrow (vi), applied to $D := D_{\Delta}$.

(i) \Rightarrow (ii): Let $X, Y \in D_{\Delta}$ and $m \in M$. Then $Y(\text{Fl}_t^X(m)) \in \Delta(\text{Fl}_t^X(m))$, and so 2.2.5 (i) \Rightarrow (iv), applied to $D = D_{\Delta}$ gives

$$(\text{Fl}_t^X)^*Y(m) = T\text{Fl}_{-t}^X(Y(\text{Fl}_t^X(m))) \subseteq \Delta(m).$$

Thus $(\text{Fl}_t^X)^*Y \in D_{\Delta}$.

(ii) \Rightarrow (iii): Set $D := D_{\Delta}$, then the claim follows from 2.2.7 (ii).

(i) \Rightarrow (iv): By 2.2.5, (i) \Rightarrow (iv), applied to D_{Δ} , it follows that $D_{\Delta} \subseteq \text{aut}(\Delta)$. Thus $\text{aut}(\Delta) \cap D_{\Delta} = D_{\Delta}$, which spans Δ by definition.

(iv) \Rightarrow (iii): Set $D := \text{aut}(\Delta) \cap D_{\Delta}$. Then D spans Δ by assumption, and for $X \in D$ and $Y \in D_{\Delta}$ we have $T\text{Fl}_{-t}^X(Y(\text{Fl}_t^X(m))) \in \Delta(m)$ since $X \in \text{aut}(\Delta)$, so $(\text{Fl}_t^X)^*Y \in D_{\Delta}$. Iterating this argument it follows that for $n \in \mathbb{N}$, $X_i, Y \in D$ ($1 \leq i \leq n$), $\xi = (X_1, \dots, X_n)$ and $T \in \mathbb{R}^n$, $(\xi_T)_*Y \in D_{\Delta}$. Therefore, $D \subseteq \mathcal{D} \subseteq D_{\Delta}$, so \mathcal{D} spans Δ and the claim follows. \square

Next, following [13, 3.25], we want to analyze the local structure of the integral manifolds of an integrable distribution. For this we first introduce an important class of immersive submanifolds:

2.2.9 Definition. Let A be any subset of M^l , and for $m \in A$ denote by $C_m(A)$ the set of all points in A that can be joined to m by a smooth curve¹ in M lying in A . A subset N of M is called an initial submanifold of M of dimension n , if for each $m \in N$ there exists a chart (U_m, φ_m) of M centered at m such that

$$\varphi_m(C_m(U_m \cap N)) = \varphi_m(U_m) \cap (\mathbb{R}^n \times \{0\}) \subseteq \mathbb{R}^n \times \mathbb{R}^{l-n}. \quad (2.2.3)$$

In order to see that calling such subsets submanifolds is justified, we need an auxiliary result:

2.2.10 Lemma. Any piecewise smooth curve $c : [a, b] \rightarrow M$ admits a reparametrization as a C^∞ -curve.

Proof. Let $a = t_0 < t_1 < \dots < t_k = b$ be such that each curve $c|_{[t_{i-1}, t_i]}$ is smooth. For each i pick a smooth map $\phi_i : [a, b] \rightarrow \mathbb{R}$ such that $\phi_i(t) = 0$ for $t \leq t_{i-1}$, $\phi_i(t) = 1$ for $t \geq t_i$, and ϕ is strictly increasing on $[t_{i-1}, t_i]$. Then the map

$$\phi := t_0 + \sum_{i=1}^k (t_i - t_{i-1}) \phi_i$$

is smooth, strictly increasing on $[a, b]$, $\phi([a, b]) = [a, b]$, $\phi(t_i) = t_i$, and $\phi^{(m)}(t_i) = 0$ for all i and all $m \geq 1$. Thus $c \circ \phi$ is the desired reparametrization. \square

2.2.11 Lemma. Under the assumptions of 2.2.9, let $m_1, m_2 \in N$ and U_{m_1}, U_{m_2} be as in (2.2.3). Then $\varphi_{m_1}(C_{m_1}(U_{m_1} \cap N) \cap C_{m_2}(U_{m_2} \cap N))$ is open in $\mathbb{R}^n \times \{0\}$.

Proof. We may suppose that the intersection is nonempty, so let $p \in C_{m_1}(U_{m_1} \cap N) \cap C_{m_2}(U_{m_2} \cap N)$. Then

$$W := C_p(U_{m_1} \cap U_{m_2} \cap N) \subseteq C_{m_1}(U_{m_1} \cap N) \cap C_{m_2}(U_{m_2} \cap N) :$$

In fact, there is a smooth curve c_1 from m_1 to p in $U_{m_1} \cap N$ and if $q \in W$ then there is also a smooth curve c_2 from p to q in $U_{m_1} \cap U_{m_2} \cap N$. The concatenation of c_1 and c_2 can be reparametrized smoothly by 2.2.10, so $q \in C_{m_1}(U_{m_1} \cap N)$, and analogously for m_2 . We claim that

$$\varphi_{m_1}(W) = C_{\varphi_{m_1}(p)}(\varphi_{m_1}(U_{m_1} \cap U_{m_2}) \cap (\mathbb{R}^n \times \{0\})). \quad (2.2.4)$$

\subseteq : $\varphi_{m_1}(W) \subseteq \varphi_{m_1}(C_{m_1}(U_{m_1} \cap N)) = \varphi_{m_1}(U_{m_1}) \cap (\mathbb{R}^n \times \{0\})$ and $\varphi_{m_1}(W) \subseteq \varphi_{m_1}(U_{m_1} \cap U_{m_2})$. Moreover, any $q \in W$ is connected to p within W by a smooth curve c , and so $\varphi_{m_1} \circ c$ connects the corresponding images.

\supseteq : Let \tilde{c} be a smooth curve from $\varphi_{m_1}(p)$ to $\varphi_{m_1}(q)$ in $\varphi_{m_1}(U_{m_1} \cap U_{m_2}) \cap (\mathbb{R}^n \times \{0\})$. Then $c := \varphi_{m_1}^{-1} \circ \tilde{c}$ is a smooth curve in $U_{m_1} \cap U_{m_2} \cap N$ from p to q .

The right hand side of (2.2.4) is precisely the connected component of $\varphi_{m_1}(p)$ in $\varphi_{m_1}(U_{m_1} \cap U_{m_2}) \cap (\mathbb{R}^n \times \{0\})$ in $\mathbb{R}^n \times \{0\}$, hence is open. Since p was arbitrary, this shows that $\varphi_{m_1}(C_{m_1}(U_{m_1} \cap N) \cap C_{m_2}(U_{m_2} \cap N))$ is open, as claimed. \square

2.2.12 Theorem. Let N be an initial submanifold of dimension n of M^l . Then there is a unique C^∞ -structure on N such that the inclusion map $i : N \hookrightarrow M$ becomes an injective immersion and such that the following universal property holds: For any manifold R^k and any map $f : R \rightarrow N$, f is smooth if and only if $i \circ f : R \rightarrow M$ is smooth. Moreover, N is paracompact.

¹By a smooth curve here we simply mean a C^∞ -map from some interval into M .

Proof. For any $m \in N$, by 2.2.9 there exists a chart (U_m, φ_m) of M around m such that $\varphi_m(C_m(U_m \cap N)) = \varphi_m(U_m) \cap (\mathbb{R}^n \times \{0\})$. We define a chart for N at m by $(C_m(U_m \cap N), \psi_m := \varphi_m|_{C_m(U_m \cap N)})$. Then the chart transition functions $\psi_{m_1} \circ \psi_{m_2}^{-1}$ are the restrictions of the smooth transition functions $\varphi_{m_1} \circ \varphi_{m_2}^{-1}$ of M to subsets of $\mathbb{R}^n \times \{0\}$ that are open by 2.2.11, hence we obtain a differentiable structure on N . From this choice of charts it immediately follows that i is an immersion. Thus N is an immersive submanifold of M . In particular, the natural manifold topology of N is finer than the trace topology of M on N .

To show (the non-trivial part of) the universal property, let $f: R \rightarrow N$ be such that $i \circ f: R \rightarrow M$ is smooth, let $r \in R$ and choose a chart (U, φ) at $f(r)$ in M such that $\varphi(C_{f(r)}(U \cap N)) = \varphi(U) \cap (\mathbb{R}^n \times \{0\})$. Then since $f^{-1}(U)$ is open in R we may pick a chart (V, ψ) in R at r with $V \subseteq f^{-1}(U)$ such that $\psi(V)$ is a ball in \mathbb{R}^k . Then since V is \mathcal{C}^∞ -contractible and $i \circ f$ is smooth it follows that any point in $f(V)$ can be connected by a smooth curve in M that lies entirely in $U \cap N$ with $f(r)$. Therefore, $f(V) \subseteq C_{f(r)}(U \cap N)$, and we can write

$$(\varphi|_{C_{f(r)}(U \cap N)}) \circ f \circ \psi^{-1} = \varphi \circ f \circ \psi^{-1}.$$

Here, the right hand side is smooth by assumption and the left hand side is the chart representation of f around r , as a map from R to N , so indeed $f: R \rightarrow N$ is \mathcal{C}^∞ .

To see uniqueness, denote by N' another differentiable structure on N with the universal property. Then $\text{id}: N \rightarrow M$ and $\text{id}: N' \rightarrow M$ are smooth, hence so are $\text{id}: N \rightarrow N'$ and $\text{id}: N' \rightarrow N$, i.e., the smooth structures in fact coincide.

Concerning the paracompactness of N , note that any connected component of N is contained in a connected component of M , hence is second countable by [11, 14.9]. Alternatively, M can be equipped with a Riemannian metric g (simply by gluing local Riemannian metrics on charts via a partition of unity) and this induces a Riemannian metric i^*g on N . The claim then follows from [7, Satz 1.1.2]. \square

Next we show that also a converse of the previous theorem holds.

2.2.13 Theorem. *Let $f: M^k \rightarrow N^l$ be an injective immersion between manifolds which has the universal property from 2.2.12, i.e.: if $h: P \rightarrow N$ is smooth and $h(P) \subseteq f(M)$, then the induced map $\bar{h}: P \rightarrow M$ with $f \circ \bar{h} = h$ is smooth. Then $f(M)$ is an initial submanifold of N .*

Proof. By [10, 1.1.3], given any $m \in M$, there exist charts (φ, W) centered at m in M and (ψ, V) centered at $f(m)$ in N such that

$$(\tilde{\psi} \circ f \circ \varphi^{-1}) = i := (x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0) \in \mathbb{R}^l.$$

We may assume that $W = f^{-1}(V)$, so $i: \varphi(W) \rightarrow \tilde{\psi}(V)$. Pick $r > 0$ so small that $\{x \in \mathbb{R}^k \mid |x| < 2r\} \subseteq \varphi(W)$ and $\{y \in \mathbb{R}^l \mid |y| < 2r\} \subseteq \tilde{\psi}(V)$. Define $B_r^k(0)$ resp. $B_r^l(0)$ to be the open ball of radius r in \mathbb{R}^k resp. \mathbb{R}^l , and set

$$U := \tilde{\psi}^{-1}(B_r^l(0)) \subseteq N, \quad W_1 := \varphi^{-1}(B_r^k(0)) \subseteq M$$

We show that $(\psi|_U, U)$ satisfies (2.2.3). In fact, since

$$\begin{aligned} i(i^{-1}(\{(y^1, \dots, y^k, 0, \dots, 0) \mid |y| < r\})) &= \{(y^1, \dots, y^k, 0, \dots, 0) \mid |y| < r\} \cap i(\varphi(W)) \\ &= \{(y^1, \dots, y^k, 0, \dots, 0) \mid |y| < r\}, \end{aligned}$$

we have

$$\begin{aligned} \psi^{-1}(\psi(U) \cap (\mathbb{R}^k \times \{0\})) &= \psi^{-1}(\{(y^1, \dots, y^k, 0, \dots, 0) \mid |y| < r\}) \\ &= f \circ \varphi^{-1}((\psi \circ f \circ \varphi^{-1})^{-1}(\{(y^1, \dots, y^k, 0, \dots, 0) \mid |y| < r\})) \\ &= f \circ \varphi^{-1}(B_r^k(0)) = f(W_1). \end{aligned}$$

Now $\tilde{\psi} \circ f(W_1) = i(B_r^k(0)) \subseteq B_r^l(0)$, so $f(W_1) \subseteq U \cap f(M)$. Also, $f(W_1)$ is \mathcal{C}^∞ -contractible, so altogether we get

$$\psi^{-1}(\psi(U) \cap (\mathbb{R}^k \times \{0\})) = f(W_1) \subseteq C_{f(m)}(U \cap f(M)).$$

Conversely, let $n \in C_{f(m)}(U \cap f(M))$. This means that there exists a smooth curve $c : [0, 1] \rightarrow N$ with $c(0) = f(m)$, $c(1) = n$, and $c([0, 1]) \subseteq U \cap f(M)$. By the universal property of f it follows that the unique curve $\bar{c} : [0, 1] \rightarrow M$ with $c = f \circ \bar{c}$ is smooth.

We show that $\bar{c}([0, 1]) \subseteq W_1$. Since $\bar{c}([0, 1]) \subseteq f^{-1}(U) \subseteq f^{-1}(V) = W$ and $\bar{c}(0) = m$ (since f is injective), it follows that $\varphi(\bar{c}(0)) = 0$. Thus if $\bar{c}([0, 1]) \not\subseteq W_1$, by continuity of \bar{c} there must exist some $t \in (0, 1]$ where $\varphi \circ \bar{c}$ intersects $\partial B_r^k(0)$, i.e., with $\bar{c}(t) \in \varphi^{-1}(\partial B_r^k(0))$. But then

$$\psi \circ f(\bar{c}(t)) \in \psi \circ f \circ \varphi^{-1}(\partial B_r^k(0)) = i(\partial B_r^k(0)) \subseteq \partial B_r^l(0),$$

and so $\psi \circ c(t) = \psi \circ f \circ \bar{c}(t) \notin B_r^l(0)$, i.e., $c(t) \notin U$, a contradiction. We conclude that $\bar{c}([0, 1]) \subseteq W_1$, and therefore $n = f(\bar{c}(1)) \in f(W_1)$. Altogether,

$$C_{f(m)}(U \cap f(M)) = f(W_1) = \psi^{-1}(\psi(U) \cap (\mathbb{R}^k \times \{0\})),$$

which shows that $f(M)$ is an initial submanifold of N . \square

The previous result in particular applies to the situation where $f = j : M \hookrightarrow N$, i.e., where M is an immersive submanifold with the universal property. In fact, the setup of 2.2.13 is only seemingly more general: if $f : M \rightarrow N$ is an injective immersion, then by transporting the manifold structure of M to $f(M)$ via f , i.e., by declaring $f : M \rightarrow f(M)$ to be a diffeomorphism it follows that $j : f(M) \hookrightarrow N$ becomes an immersion, hence $f(M)$ turns into an immersive submanifold of N , and the universal property from 2.2.13 translates into the one from 2.2.12.

We now return to the study of integrable distributions.

2.2.14 Theorem. *Let Δ be an integrable distribution on M^l . Let S be a non-trivial orbit of D_Δ and let $m \in S$. Then there exists a cubic chart $(U, \varphi = (x_1, \dots, x_l))$ centered at m , $\varphi(U) = (-\varepsilon, \varepsilon)^l$ for some $\varepsilon > 0$, some $k \geq 1$ and a countable set $A \subseteq \mathbb{R}^{l-k}$ such that*

$$\varphi(U \cap S) = \{x \in \varphi(U) \mid (x^{k+1}, \dots, x^l) \in A\}.$$

If the distribution is of constant rank k then the above holds for every non-trivial orbit intersecting U , with the same k . Moreover, each non-trivial orbit is an initial submanifold of M .

Proof. Let $k := \dim(S)$ and pick $X_1, \dots, X_k \in D_\Delta$ such that $\{X_1(m), \dots, X_k(m)\}$ is a basis of $\Delta(m)$. Next, choose a chart $(\chi = (y^1, \dots, y^l), W)$ around m in M such that $X_1(m), \dots, X_k(m), \frac{\partial}{\partial y^{k+1}} \Big|_m, \dots, \frac{\partial}{\partial y^l} \Big|_m$ is a basis of $T_m M$. Let

$$f(t^1, \dots, t^l) := (\text{Fl}_{t^1}^{X_1} \circ \dots \circ \text{Fl}_{t^k}^{X_k})(\chi^{-1}(0, \dots, 0, t^{k+1}, \dots, t^l)).$$

Then f is a diffeomorphism from some neighborhood of $0 \in \mathbb{R}^l$ onto a neighborhood of m in M , and we take $\varphi := f^{-1}$ on a suitable neighborhood U of m , for which we may suppose that $\varphi(U)$ is a cube $(-\varepsilon, \varepsilon)^l$ with center $0 = \varphi(m)$. Since S is an orbit of D_Δ ,

$$m' \in S \Leftrightarrow \text{Fl}_{t^1}^{X_1} \circ \dots \circ \text{Fl}_{t^k}^{X_k}(m') \in S$$

for all m' and t^1, \dots, t^k where the right hand side is defined. Therefore, for any $m' = f(t^1, \dots, t^l) \in U$ we have

$$m' = f(t^1, \dots, t^l) \in S \Leftrightarrow f(0, \dots, 0, t^{k+1}, \dots, t^l) \in S. \quad (2.2.5)$$

This means that $U \cap S$ is the disjoint union of connected sets of the form

$$U_c := \{m' \in U \mid x^{k+1}(m') = c_{k+1}, \dots, x^l(m') = c_l\}$$

where $c = (c_{k+1}, \dots, c_l)$ is constant. By assumption, Δ is integrable, so 2.2.5 and 2.2.4 show that any orbit is a maximal integral manifold of Δ . Therefore, since

$$U_c = \{(\text{Fl}_{t^1}^{X_1} \circ \dots \circ \text{Fl}_{t^k}^{X_k})(\chi^{-1}(0, c)) \mid (t^1, \dots, t^k) \in (-\varepsilon, \varepsilon)^k\},$$

and S is an integral manifold of Δ , the proof of 2.2.3 demonstrates that U_c is an open (and connected) submanifold of S . Now S , being a connected immersive submanifold, is contained in a connected component C of M , and C is second countable since M is paracompact. Thus by [11, 14.9], S is itself second countable. This shows that there can at most be countably many U_c as above. If Δ is of constant rank k then clearly the above construction works for this same k for any orbit that intersects U .

Finally, from (2.2.5) it follows that $m' \in C_m(U \cap S)$ if and only if $m' \in U_0$, i.e.,

$$\varphi(C_m(U \cap S)) = \varphi(U) \cap (\mathbb{R}^k \times \{0\}),$$

so S is indeed an initial submanifold of M . □

2.2.15 Remark. The previous result provides an alternative proof of 2.1.1, (iii) \Leftrightarrow (iv).

2.2.16 Definition. A chart as in 2.2.14 is called a distinguished chart for Δ . The connected components of $U \cap S$ are called plaques (or slices).

2.2.17 Definition. Let $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ be everywhere defined. D is said to satisfy the reachability condition if the D -orbits are exactly the connected components of M .

A necessary and sufficient condition for reachability is given in the following result:

2.2.18 Theorem. Let $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ be everywhere defined. Then the following are equivalent:

- (i) D satisfies the reachability condition.
- (ii) For every $m \in M$ we have $\dim P_D(m) = \dim(M)$.

Proof. Let $n := \dim(M)$.

(i) \Rightarrow (ii) Suppose that for some $m \in M$ we have $k := \dim P_D(m) < n$. Then (by 2.2.4 (i)) the orbit S of D through m is a k -dimensional connected immersive submanifold of M . It follows that the interior of S in the topology of M is empty. But any connected component of M is open, so S cannot be such a connected component, contradicting our assumption (i).

(ii) \Rightarrow (i) By [11, 14.1], every maximal integral manifold of P_D is open (being an immersive submanifold of the same dimension as M), and connected. Also, M is

the disjoint union of the orbits of D which, by 2.2.4 (ii), are exactly the maximal integral manifolds of P_D . Thus these orbits are the connected components of M . \square

Recall from 2.1.14 that if D is any subset of $\mathfrak{X}_{\text{loc}}(M)$ then the smallest involutive subset of $\mathfrak{X}_{\text{loc}}(M)$ that contains D is denoted by D^* .

2.2.19 Corollary. *With M, n, D as in 2.2.18, if Δ_{D^*} has dimension n for every $m \in M$ then D satisfies the reachability condition.*

Proof. By 2.1.15, $\Delta_{D^*} \subseteq P_D$, so $\dim(P_D(m)) = n$ for all $m \in M$, and the claim follows from 2.2.18. \square

To conclude this chapter we show how the classical results on the integrability of constant rank distributions can be derived from the results established above.

2.2.20 Definition. *Let $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ be everywhere defined. D is called locally of finite type if for every $m \in M$ there exist $X^1, \dots, X^k \in D$ such that*

- (i) $\Delta_D(m) = \text{span}(X^1(m), \dots, X^k(m))$, and
- (ii) *for every $X \in D$ there exists a neighborhood U of m in M and smooth functions $f_j^i \in \mathcal{C}^\infty(U)$ ($1 \leq i, j \leq k$) such that*

$$[X, X^i](m') = \sum_{j=1}^k f_j^i(m') X^j(m')$$

for $i = 1, \dots, k$ and all $m' \in U$.

2.2.21 Theorem. *If D is locally of finite type then Δ_D has the maximal integral manifold property.*

Proof. By assumption, $\Delta = \Delta_D$ satisfies (v), and thereby also (ii) of 2.2.5. \square

2.2.22 Corollary. *If D^* is locally of finite type then $\Delta_{D^*} = P_D$ and the non-trivial D -orbits are precisely the maximal integral manifolds of Δ_{D^*} .*

Proof. We know from 2.1.15 that $\Delta_{D^*} \subseteq P_D$. By 2.2.21, Δ_{D^*} has the maximal integral manifold property, so 2.2.5 shows that Δ_{D^*} is $(D^*$ -, hence) D -invariant. Therefore by definition of P_D we get $P_D \subseteq \Delta_{D^*}$, i.e., $\Delta_{D^*} = P_D$. The claim then follows from 2.2.4 (ii). \square

We can now derive the following version of the classical Frobenius theorem.

2.2.23 Theorem. (Frobenius) *Let Δ be a smooth distribution on M of constant rank k . Then the following are equivalent:*

- (i) Δ has the maximal integral manifold property.
- (ii) Δ is involutive.

Proof. (i) \Rightarrow (ii) This is 2.1.18.

(ii) \Rightarrow (i) Let $m \in M$ and pick $X^1, \dots, X^k \in \Delta$ such that $\{X^1(m), \dots, X^k(m)\}$ is a basis of $\Delta(m)$. Then also $\{X^1(m'), \dots, X^k(m')\}$ is linearly independent for all m' in some neighborhood U of m in M . Since $\dim(\Delta(m')) = k$ for all m' , any local vector field on U that belongs to Δ is a linear combination of X^1, \dots, X^k with

smooth coefficients. If X belongs to Δ then since Δ is involutive, $[X, X^i] \in \Delta$, and so D_Δ is locally of finite type. Since $\Delta_{D_\Delta} = \Delta$ by definition, (i) follows from 2.2.21. \square

2.2.24 Corollary. *Let $D \subseteq \mathfrak{X}_{\text{loc}}(M)$ be everywhere defined, involutive and of constant rank. Then Δ_D has the maximal integral property.*

Proof. By 2.2.23 it suffices to show that Δ_D is involutive. Let $m \in M$ and pick $X^1, \dots, X^k \in \Delta$ such that $\{X^1(m'), \dots, X^k(m')\}$ is linearly independent for all m' in some neighborhood U of m in M . If $X, X' \in \Delta_D$ on U then both X and X' are linear combinations of X^1, \dots, X^k with smooth coefficients. But then also $[X, X']$ is such a linear combination of the X^j and of brackets of X^j and X^l , which also belong to Δ_D due to the involutivity of D . Thus $[X, X'] \in \Delta_D$. \square

2.2.25 Remark. Collecting some of the results proved above we obtain an independent proof of the classical Frobenius theorem 2.1.1. In fact, (iii) \Leftrightarrow (iv) follows from 2.2.15. 2.2.23 shows that (i) \Rightarrow (iii), and (iii) \Rightarrow (i) is clear. (ii) \Rightarrow (i) is the easy part of [11, 17.13].

Finally, to see that (iv) \Rightarrow (ii), let (φ, U) be a chart as in (iv). Then for any point m in U , $\partial_{x^1}|_m, \dots, \partial_{x^k}|_m$ span the tangent space at m of the slice U_a containing m . But U_a is an integral manifold of Δ , so this tangent space equals $\Delta(m)$.

Chapter 3

Symmetry groups of differential equations

3.1 Local transformation groups

In Chapter 1 we studied Lie transformation groups on differentiable manifolds. Such group actions are always defined globally ($\Phi: G \times M \rightarrow M$). For the applications to symmetry groups of differential equations we have in mind, the natural actions will typically not be defined globally, however. In this section, following [17, 14], we therefore study local transformation groups by means of the tools developed in Chapter 2. Throughout, we will assume M to be a connected paracompact (hence Hausdorff and second countable) \mathcal{C}^∞ -manifold.

3.1.1 Definition. *A local transformation group (or local Lie transformation group) on M consists of a Lie group G , an open subset \mathcal{U} of $G \times M$ with $\{e\} \times M \subseteq \mathcal{U}$, and a smooth map $\Phi: \mathcal{U} \rightarrow M$ such that*

(i) *If $(h, m) \in \mathcal{U}$, $(g, \Phi(h, m)) \in \mathcal{U}$ and $(gh, m) \in \mathcal{U}$ then*

$$\Phi(g, \Phi(h, m)) = \Phi(gh, m).$$

(ii) *For all $m \in M$, $\Phi(e, m) = m$.*

(iii) *If $(g, m) \in \mathcal{U}$ then $(g^{-1}, \Phi(g, m)) \in \mathcal{U}$ and (by (i), (ii))*

$$\Phi(g^{-1}, \Phi(g, m)) = m.$$

We will often abbreviate $\Phi(g, m)$ by $g \cdot m$. We set

$$\begin{aligned}\mathcal{U}_g &:= \{m \in M \mid (g, m) \in \mathcal{U}\} \quad (g \in G) \\ \mathcal{U}_m &:= \{g \in G \mid (g, m) \in \mathcal{U}\} \quad (m \in M)\end{aligned}$$

and

$$\begin{aligned}\Phi_g: \mathcal{U}_g &\rightarrow M, \quad m \mapsto \Phi(g, m) \\ \Phi_m: \mathcal{U}_m &\rightarrow M, \quad g \mapsto \Phi(g, m)\end{aligned}$$

For $\mathcal{U} = G \times M$ we obtain a global transformation group as in 1.1.2.

3.1.2 Definition. A subset $\emptyset \neq S \subseteq M$ is called an orbit of the local transformation group $\Phi: \mathcal{U} \subseteq G \times M \rightarrow M$, or a Φ -orbit, if it is a minimal G -invariant subset of M , i.e.,

(i) If $m \in S$, $g \in G$ and $(g, m) \in \mathcal{U}$ then $\Phi(g, m) \in S$.

(ii) If $S' \subseteq S$ is another subset of M satisfying (i) then either $S' = \emptyset$ or $S' = S$.

For any $m \in M$ we denote by $S_{\Phi, m}$ the orbit of m under Φ .

3.1.3 Remark. If Φ is a global transformation group then

$$S_{\Phi, m} = \Phi_m(G) = G \cdot m = \{\Phi(g, m) \mid g \in G\}.$$

For a local transformation group $\Phi: \mathcal{U} \rightarrow M$ we obtain

$$S_{\Phi, m} = \{p \in M \mid \exists k \in \mathbb{N}, \exists g_i \in G \ (1 \leq i \leq k) : g_k \in \mathcal{U}_m, \\ g_i \in \mathcal{U}_{g_{i+1} \cdots g_k \cdot m} \ (1 \leq i \leq k-1) \text{ and } g_1 \cdots g_k \cdot m = p\} \quad (3.1.1)$$

In what follows, we will illustrate many of the concepts we consider in the following example:

3.1.4 Example. Let $M = \mathbb{R}^2$, $G = (\mathbb{R}, +)$, and

$$\Phi(\varepsilon, (x, y)) := \left(\frac{x}{1 - \varepsilon x}, \frac{y}{1 - \varepsilon x} \right).$$

The natural domain of Φ is

$$\mathcal{U} = \{(\varepsilon, (x, y)) \mid \varepsilon < \frac{1}{x} \text{ for } x > 0, \varepsilon > \frac{1}{x} \text{ for } x < 0, \varepsilon \in \mathbb{R} \text{ for } x = 0\} \subseteq \mathbb{R} \times \mathbb{R}^2,$$

which is an open subset of $G \times M$, and Φ is \mathcal{C}^∞ on \mathcal{U} . One easily checks that

$$\Phi(\varepsilon_1, \Phi(\varepsilon_2, (x, y))) = \Phi(\varepsilon_1 + \varepsilon_2, (x, y)),$$

whenever both sides are defined. Thus we obtain a local transformation group. Its orbits are the points on the y -axis and the straight half-rays emanating from the origin (except for the positive and negative y -axis). Thus they are either single points or (regular) submanifolds of M .

Note that Φ cannot be realized as the restriction to \mathcal{U} of some global Lie transformation group on \mathbb{R}^2 : in fact, for any $x \neq 0$ we have $|\Phi(\varepsilon, (x, y))| \rightarrow \infty$ as $\varepsilon \rightarrow \frac{1}{x}$.

3.1.5 Definition. A local transformation group $\Phi: G \times M \supseteq \mathcal{U} \rightarrow M$ is called

(i) connected, if

(a) M and G are connected.

(b) \mathcal{U} is connected.

(c) \mathcal{U}_m is connected for each $m \in M$.

(ii) semi-regular, if all orbits can be endowed with a smooth structure as immersive submanifolds of M of the same dimension.

(iii) regular, if it is semi-regular and every $m \in M$ possesses a neighborhood basis of open sets U such that for every orbit S of G the set $U \cap S$ is connected in S .

Blanket assumption: From now on we will assume all local transformation groups to be connected in the above sense.

3.1.6 Remark.

- (i) Since S (being a manifold) is locally pathwise connected, a semi-regular transformation group is regular if and only if every $m \in M$ possesses a neighborhood basis of open sets U such that for every orbit S of G the set $U \cap S$ is pathwise connected in S .
- (ii) By [10, 1.1.12], every orbit of a regular transformation group is a regular submanifold of M .

3.1.7 Example. (i) The map Φ from 3.1.4 defines a regular transformation group on $\mathbb{R}^2 \setminus \{(0, y) \mid y \in \mathbb{R}\}$.

(ii) (Cf. [11, Ex. 18.7]) Let $M = T^2 = S^1 \times S^1$ be the two-dimensional torus and $G = (\mathbb{R}, +)$. Fixing $\omega \in \mathbb{R}$ and using angular coordinates (θ, ρ) on M we set

$$\Phi(\varepsilon, (\theta, \rho)) := (\theta + \varepsilon, \rho + \omega\varepsilon) \mod 2\pi.$$

Then the orbits of Φ are immersive submanifolds of dimension 1, so G acts semi-regularly on M . If $\omega \in \mathbb{Q}$ then the orbits are closed curves and Φ acts regularly. However, if ω is irrational then the orbits are dense in M and therefore cannot be regular submanifolds by [11, 14.1]. So in this case Φ does not act regularly on M .

Our next aim is to show that the (non-discrete) orbits of any local transformation group can naturally be endowed with the structure of an immersive (indeed even initial) submanifold of M .

3.1.8 Remark. If $\Phi: G \times M \rightarrow M$ is a global (connected) Lie transformation group on M then we have already shown this in 1.2.4 and 1.2.5: Denoting by G_m the isotropy group of m in G we have by 1.2.2 that the map $\Psi_m: G/G_m \rightarrow G \cdot m$, $gG_m \mapsto gm$ is a bijection. If the isotropy group G_m of m is open then by 1.2.6 the orbit of m is the singleton $\{m\}$. Otherwise, declaring Ψ_m to be a diffeomorphism we may endow $G \cdot m$ with a smooth structure as an immersive submanifold of M . The fact that $G \cdot m$ is in fact an initial submanifold of M will follow from 3.1.14 and 3.1.15 below.

Turning now to the case of local transformation groups, we need to come up with a different construction since in general G_m will no longer be a subgroup of G in this case: in fact, if $g_1, g_2 \in G_m$ then in general we will not have that $(g_1^{-1} \cdot g_2, m) \in \mathcal{U}$. But this fact was used in 1.2.2 to obtain injectivity of Ψ_m .

The route we will take to finding a smooth structure on the orbits of G goes via the infinitesimal generators of the action of Φ . We first note that also for a local transformation group we may introduce the definitions of $\Phi(v)$ and $\mathcal{R}(G, M)$ exactly as in (1.1.2). However, $\mathcal{R}(G, M) \subseteq \mathfrak{X}_{\text{loc}}(M)$ now. Also, as in 1.1.12 it follows that the map $\Phi: v \mapsto \Phi(v)$ is a Lie algebra homomorphism from \mathfrak{g}_R onto $\mathcal{R}(G, M)$, the Killing algebra of Φ . Concerning 1.1.13 we have to be careful about the domain \mathcal{U}_m and note that for a local transformation group the vector field $\Phi(v)$ need no longer be complete:

3.1.9 Proposition. *Let $\Phi: \mathcal{U} \rightarrow M$ be a local transformation group, and let $m \in M$ and $v \in \mathfrak{g}$. Then for all ε such that $\exp(\varepsilon v) \in \mathcal{U}_m$ we have*

$$\text{Fl}_\varepsilon^{\Phi(v)}(m) = \Phi(\exp(\varepsilon v), m) = \exp(\varepsilon v) \cdot m.$$

In particular,

$$\Phi(v)(m) = \left. \frac{d}{d\varepsilon} \right|_0 \Phi(\exp(\varepsilon v), m).$$

Proof. For ε in a sufficiently small interval around 0, $\exp(\varepsilon v) \in \mathcal{U}_m$. On any such interval (hence on the maximal such interval), we can argue as in the proof of 1.1.13. \square

For any $m \in M$ we set

$$\mathcal{R}_m(G, M) := \{\Phi(v)(m) \mid v \in \mathfrak{g}\}.$$

Any $\mathcal{R}_m(G, M)$ is a linear subspace of $T_m M$.

3.1.10 Example. For the example from 3.1.4 we have

$$\begin{aligned} \Phi(\partial_\varepsilon|_0)(x, y) &= T_0\Phi_{(x,y)}(\partial_\varepsilon|_0) = \partial_\varepsilon|_0 \left(\frac{x}{1-\varepsilon x} \right) \partial_x + \partial_\varepsilon|_0 \left(\frac{y}{1-\varepsilon x} \right) \partial_y \\ &= x^2 \partial_x + xy \partial_y =: X(x, y). \end{aligned}$$

Therefore, $\mathcal{R}_{(x,y)}(G, M)$ is one-dimensional if $x \neq 0$ and equals $\{0\}$ for $x = 0$.

To verify 1.1.13 in this example (cf. 1.1.14) we first calculate the flow of X . The integral curve $c(\varepsilon) = (x(\varepsilon), y(\varepsilon))$ of X through (x_0, y_0) has to satisfy the following initial value problem:

$$\begin{aligned} \partial_\varepsilon x(\varepsilon) &= x(\varepsilon)^2 \\ \partial_\varepsilon y(\varepsilon) &= x(\varepsilon)y(\varepsilon) \\ (x(0), y(0)) &= (x_0, y_0). \end{aligned}$$

Indeed we obtain

$$(x(\varepsilon), y(\varepsilon)) = \left(\frac{x_0}{1-\varepsilon x_0}, \frac{y_0}{1-\varepsilon x_0} \right) = \Phi(\varepsilon, (x_0, y_0))$$

with maximal domain $(-\infty, \frac{1}{x_0})$ if $x_0 > 0$, $(\frac{1}{x_0}, \infty)$ if $x_0 < 0$, and \mathbb{R} , respectively, if $x_0 = 0$, i.e., the maximal domain of c is precisely $\mathcal{U}_{(x_0, y_0)}$. In general, one can only expect that $\mathcal{U}_{(x_0, y_0)}$ is contained in the domain of the corresponding maximal integral curve. Also, the image of c is the orbit of (x_0, y_0) .

3.1.11 Proposition. Let $\Phi: G \times M \supseteq \mathcal{U} \rightarrow M$ be a local transformation group. Then the Killing algebra $\mathcal{R}(G, M)$ of Φ spans an integrable distribution

$$\Delta_\Phi := \Delta_{\mathcal{R}(G, M)}$$

on M .

Proof. By definition (see (1.1.3)), $\mathcal{R}(G, M)$ consists of the smooth local vector fields $\Phi(v)$ for $v \in \mathfrak{g}$, hence Δ_Φ is a smooth distribution on M . Pick any basis $\{v_1, \dots, v_k\}$ of \mathfrak{g} and set $X_i := \Phi(v_i) \in \mathcal{R}(G, M)$ for $1 \leq i \leq k$. Then by 1.1.12, $\{X_1, \dots, X_k\}$ is a Lie algebra of local vector fields spanning Δ_Φ . Consequently, Δ_Φ satisfies condition (v) of 2.2.5 and thereby is integrable. \square

Note that the dimension of Δ_Φ may vary from point to point, so we really need the theory of distributions of non-constant rank from Chapter 2. From 3.1.11 it follows by 2.2.4 and 2.2.14 that the non-trivial orbits $S_{\Delta_\Phi, m}$ of Δ_Φ are initial submanifolds of M . It therefore remains to show that these orbits in fact coincide with those introduced in 3.1.2, i.e., that $S_{\Phi, m} = S_{\Delta_\Phi, m}$. For this we need some auxiliary results.

3.1.12 Lemma. *Let $\Phi: \mathcal{U} \rightarrow M$ be a local transformation group and as in 3.1.1, for $m \in M$ let $\mathcal{U}_m = \{g \in G \mid (g, m) \in \mathcal{U}\}$ ($m \in M$). Then for any $g \in \mathcal{U}_m$ there exist $n \in \mathbb{N}$ and $v_1, \dots, v_n \in \mathfrak{g}$, $t_1, \dots, t_n \in \mathbb{R}_{\geq 0}$ such that*

$$(i) \quad g = \exp(t_1 v_1) \cdots \exp(t_n v_n)$$

$$(ii) \quad \forall s \in [0, t_n] : \exp(s v_n) \in \mathcal{U}_m$$

$$(iii) \quad \forall i = 1, \dots, n-1 \quad \forall s \in [0, t_i] : \exp(s v_i) \exp(t_{i+1} v_{i+1}) \cdots \exp(t_n v_n) \in \mathcal{U}_m.$$

Proof. Denote by \mathcal{W} the set of all $g \in \mathcal{U}_m$ that satisfy (i)–(iii). Then $\mathcal{W} \neq \emptyset$ since $e \in \mathcal{W}$. To show that \mathcal{W} is open, let $g \in \mathcal{W}$ and pick an absolutely convex neighborhood V of 0 in \mathfrak{g} with $\exp(V)g \subseteq \mathcal{U}_m$. Then $\exp(V)g \subseteq \mathcal{W}$: given $h \in \exp(V)g$ we can write

$$h = \exp(v) \exp(t_1 v_1) \cdots \exp(t_n v_n),$$

and since V is absolutely convex it follows that also $\exp(tv) \exp(t_1 v_1) \cdots \exp(t_n v_n) \in \mathcal{U}_m$ for all $t \in [0, 1]$. But also $\mathcal{U}_m \setminus \mathcal{W}$ is open: suppose that $g \in \mathcal{U}_m \setminus \mathcal{W}$ and again choose an absolutely convex neighborhood V of 0 in \mathfrak{g} with $\exp(V)g \subseteq \mathcal{U}_m$. Suppose that $(\exp(V)g) \cap \mathcal{W} \neq \emptyset$, then there exists some $v \in V$ such that $h := \exp(v)g \in \mathcal{W}$. Since $\exp(V)g \subseteq \mathcal{U}_m$ we also have $\exp(tv)g \in \mathcal{U}_m$ for all $|t| \leq 1$. It follows that

$$g_t := \exp(t(-v))h = \exp((1-t)v)g \in \mathcal{U}_m \quad (t \in [0, 1]).$$

This, however, shows that $g = g_1$ is an element of \mathcal{W} , contradicting our assumption. Therefore, $\exp(V)g \subseteq \mathcal{U}_m \setminus \mathcal{W}$, implying that $\mathcal{U}_m \setminus \mathcal{W}$ is open. Since \mathcal{U}_m is connected by our blanket assumption following 3.1.5, $\mathcal{W} = \mathcal{U}_m$. \square

3.1.13 Lemma. *Let $\Phi: \mathcal{U} \rightarrow M$ be a local transformation group, let $m \in M$, $g \in \mathcal{U}_m$ and pick $n \in \mathbb{N}$, $v_1, \dots, v_n \in \mathfrak{g}$, $t_1, \dots, t_n \in \mathbb{R}_{\geq 0}$ such that (i)–(iii) of 3.1.12 are satisfied. Define curves*

$$\gamma_l : [0, \sum_{i=1}^n t_i] \rightarrow G \quad (l = 1, 2)$$

for $s \in [0, t_n]$ by

$$\gamma_1(s) := \exp(s v_n) m \quad \text{and} \quad \gamma_2(s) := \text{Fl}_s^{\Phi(v_n)}(m),$$

and for $s \in [\sum_{j=i+1}^n t_j, \sum_{j=i}^n t_j]$ ($i = 1, \dots, n-1$) by

$$\begin{aligned} \gamma_1(s) &:= \left(\exp \left(\left(s - \sum_{j=i+1}^n t_j \right) v_i \right) \exp(t_{i+1} v_{i+1}) \cdots \exp(t_n v_n) \right) \cdot m \\ \gamma_2(s) &:= \text{Fl}_{s - \sum_{j=i+1}^n t_j}^{\Phi(v_i)} (\text{Fl}_{t_{i+1}}^{\Phi(v_{i+1})} (\cdots \text{Fl}_{t_n}^{\Phi(v_n)}(m)) \cdots). \end{aligned}$$

Then $\gamma_1(s) = \gamma_2(s)$ for all $s \in [0, \sum_{i=1}^n t_i]$.

Proof. For $s \in [0, t_n]$ this is immediate from 3.1.12 and 3.1.9. Proceeding by induction, suppose that we already know that $\gamma_1 = \gamma_2$ on $[0, \sum_{j=i+1}^n t_j]$. Now set

$$\begin{aligned} \rho_1(s) &:= (\exp(s v_i) \exp(t_{i+1} v_{i+1}) \cdots \exp(t_n v_n)) \cdot m \\ \rho_2(s) &:= \text{Fl}_s^{\Phi(v_i)} (\text{Fl}_{t_{i+1}}^{\Phi(v_{i+1})} (\cdots \text{Fl}_{t_n}^{\Phi(v_n)}(m)) \cdots). \end{aligned}$$

Then it suffices to show that $\rho_1 = \rho_2$ on $[0, t_i]$. We do this by showing that ρ_1 satisfies the same initial value problem as ρ_2 on $[0, t_i]$. Set $\exp(t_{i+1}v_{i+1}) \dots \exp(t_nv_n) =: h$. By [11, 8.2], $R_h(\exp(sv_i)) = \text{Fl}_s^{R^{v_i}}(h)$, so

$$\frac{d}{ds}(R_h(\exp(sv_i))) = R_s^{v_i}(R_h(\exp(sv_i))).$$

Using this, together with (1.1.4), we calculate

$$\begin{aligned} \frac{d}{ds}\rho_1(s) &= \frac{d}{ds}\Phi_m(R_h(\exp(sv_i))) = T\Phi_m(R_s^{v_i}(R_h(\exp(sv_i)))) \\ &= \Phi(v_i)(\Phi_m(R_h(\exp(sv_i)))) = \Phi(v_i)(\rho_1(s)), \end{aligned}$$

which is precisely the defining ODE for ρ_2 . Also, the initial values coincide since $\rho_1(0) = \rho_2(0)$ by our inductive assumption. \square

After these preparations we may now show the equality of the two kinds of orbits we introduced above:

3.1.14 Proposition. *Let $\Phi: G \times M \supseteq \mathcal{U} \rightarrow M$ be a local transformation group and let $m \in M$. Then*

$$S_{\Phi, m} = S_{\Delta_\Phi, m}.$$

In particular, each non-trivial orbit is an initial submanifold of M .

Proof. Since G is connected, once we choose a basis $\{v_1, \dots, v_k\}$ of \mathfrak{g} , any element of G is a product of certain $\exp(\varepsilon_i v_i)$. Thus by 3.1.13 it follows that $S_{\Phi, m}$ is trivial if and only if $\Phi(v_i)(m) = 0$ for all i , i.e., if and only if $\Delta_\Phi(m) = 0$. Since Δ_Φ is integrable by 3.1.11, 2.2.5 (vi) gives that $\Delta_\Phi = P_{\mathcal{R}(G, M)}$ (recall that $\Delta_\Phi = \Delta_{\mathcal{R}(G, M)}$ by definition). Consequently, 2.1.12 shows that the trivial orbits of $S_{\Phi, m}$ and $S_{\Delta_\Phi, m}$ coincide.

Turning now to the case of non-trivial orbits, let $m' \in S_{\Delta_\Phi, m}$ and let $g \in \mathcal{U}_{m'}$. By 3.1.12 there exist $n \in \mathbb{N}$, $v_1, \dots, v_n \in \mathfrak{g}$ and $t_1, \dots, t_n \in \mathbb{R}_{\geq 0}$ such that $g = \exp(t_1 v_1) \dots \exp(t_n v_n)$ and (i)-(iii) of 3.1.12 are satisfied (with m' instead of m). Let $\xi := (\Phi(v_1), \dots, \Phi(v_n)) \in \mathfrak{X}_{\text{loc}}(M)^n$, and $T := (t_1, \dots, t_n)$. Then using the notation (2.1.1), 3.1.13 shows that $g \cdot m' = \xi_T(m')$, and by definition $\xi_T(m')$ stays in the orbit $S_{\Delta_\Phi, m}$ of m , hence $g \cdot m' \in S_{\Delta_\Phi, m}$. From this, starting with $m' = m$ and then continuing inductively, it follows that any $g_1 \dots g_l \cdot m$ stays in $S_{\Delta_\Phi, m}$, so by (3.1.1) $S_{\Phi, m} \subseteq S_{\Delta_\Phi, m}$. Since in the above consideration we started out with any $m' \in S_{\Delta_\Phi, m}$ and since the Φ -orbits are disjoint by definition we have even shown that $S_{\Delta_\Phi, m}$ is the disjoint union of certain Φ -orbits.

To conclude the proof we show that any Φ -orbit that is contained in $S_{\Delta_\Phi, m}$ is in fact an open subset of $S_{\Delta_\Phi, m}$. Indeed, once we know this then due to $S_{\Delta_\Phi, m}$ being a disjoint union of such sets it will follow that any Φ -orbit contained in $S_{\Delta_\Phi, m}$ is both open and closed in it, and since $S_{\Delta_\Phi, m}$ is connected the orbits must in fact coincide.

Thus let $m' \in S_{\Phi, m} \subseteq S_{\Delta_\Phi, m}$ (since we may move points within orbits this is the only case we need to consider). Choose $X_i = \Phi(v_i) \in \mathcal{R}(G, M)$ ($1 \leq i \leq n$, for a suitable n and suitable elements v_i of \mathfrak{g}) such that X_1, \dots, X_n is a basis for $\Delta_\Phi(m')$. Set $\xi := (X_1, \dots, X_n)$. Then by the proof of 2.2.4 (i) and 2.2.5 (vi) we have

$$\text{rk}(T_0 \rho_{\xi, m'}) = \dim(\Delta_\Phi(m')) = n.$$

As in (2.2.2) it follows that there exists an open neighborhood U of 0 in \mathbb{R}^n such that $\rho_{\xi, m'}(U)$ is the domain of a coordinate chart of $S_{\Delta_\Phi, m}$.

There exists an absolutely convex open neighborhood $V \subseteq U$ of 0 in \mathbb{R}^n such that for any $T = (t_1, \dots, t_n) \in V$ we have

$$\exp(t_1 v_1) \dots \exp(t_n v_n) \in \mathcal{U}_{m'}.$$

In particular, (i)-(iii) of 3.1.12 are satisfied. Since $\rho_{\xi, m'}(V)$ is open in $S_{\Delta_\Phi, m}$ it suffices to show that $\rho_{\xi, m'}(V) \subseteq S_{\Phi, m}$. Also, because V is absolutely convex we only need to show that $\rho_{\xi, m'}(T) \in S_{\Phi, m}$ for any $T = (t_1, \dots, t_n) \in V \cap \mathbb{R}_{\geq 0}^n$ (otherwise replace v_i by $-v_i$). For such a T , 3.1.13 (together with (2.1.1)) gives

$$\begin{aligned} \rho_{\xi, m'}(T) &= \text{Fl}_{t_1}^{X_1}(\text{Fl}_{t_2}^{X_2}(\dots \text{Fl}_{t_n}^{X_n}(m') \dots)) \\ &= \exp(t_1 v_1) \cdot \dots \cdot \exp(t_n v_n) \cdot m' \in S_{\Phi, m}, \end{aligned} \quad (3.1.2)$$

as desired. The final claim was already shown in the remark following 3.1.11. \square

If the transformation group G is in fact global then at the moment we have two ways of endowing the non-trivial orbits S_Φ of Φ with a differentiable structure: the one from 3.1.8 and the one from 3.1.11. The following result shows that these approaches in fact coincide.

3.1.15 Proposition. *Let $\Phi: G \times M \rightarrow M$ be a global Lie transformation group and let $m \in M$. Then the differentiable structures on any non-trivial orbit $S_{\Phi, m}$ introduced in 3.1.8 and in 3.1.11, 3.1.14 coincide.*

Proof. For brevity, we set $S := S_{\Phi, m}$, with the smooth structure from 3.1.11, and we write S' for the smooth structure on $S_{\Phi, m}$ defined in 3.1.8. We know that the inclusions $i: S \hookrightarrow M$ and $i': S' \hookrightarrow M$ are injective immersions. Since S is an initial submanifold of M and $i' = i \circ \text{id}_{S' \rightarrow S}$ is smooth it follows from the universal property in 2.2.12 that $\text{id}: S' \rightarrow S$ is smooth.

Conversely, to see that $\text{id}: S \rightarrow S'$ is smooth, let $\rho_{\xi, m'}(V)$ (with $m' \in S$) be as in the proof of 3.1.14. Then $\rho_{\xi, m'}: V \rightarrow \rho_{\xi, m'}(V)$ is a diffeomorphism onto an open submanifold (containing m') of S , so we only need to show that $\text{id}_{S \rightarrow S'} \circ \rho_{\xi, m'}$ is smooth on V . By 3.1.8, this is the case if and only if $\Psi_{m'}^{-1} \circ \rho_{\xi, m'}$ is smooth on V . Now since Φ is global, it follows from 1.1.13 that for any $T \in V$ we have

$$\begin{aligned} \Psi_{m'}^{-1} \circ \rho_{\xi, m'}(T) &= \Psi_{m'}^{-1}(\exp(t_1 v_1) \cdot \dots \cdot \exp(t_n v_n) \cdot m') \\ &= \exp(t_1 v_1) \cdot \dots \cdot \exp(t_n v_n) \cdot G_{m'} \end{aligned}$$

and this map is indeed smooth as the composition of the smooth maps $T \mapsto \exp(t_1 v_1) \cdot \dots \cdot \exp(t_n v_n)$, $V \rightarrow G$, and the quotient map $\pi: G \rightarrow G/G_{m'}$. \square

3.1.16 Remark. Suppose that $\Phi: G \times M \supseteq \mathcal{U} \rightarrow M$ is a semi-regular local transformation group. Then since all orbits of Φ are immersive submanifolds of the same dimension, say k , it follows that the integrable distribution Δ_Φ is of constant rank $k \geq 1$. If S is an orbit of Φ (i.e., by 3.1.14, an orbit of D_{Δ_Φ}) then by 2.2.4 (ii) S is a maximal integral manifold of Δ_Φ and thereby a leaf of Δ_Φ (cf. [11, Def. 17.23]).

Recall from [11, Def. 17.27] that a flat chart for a distribution Δ is called regular if every leaf of Δ that intersects it does so in precisely one slice. Moreover, Δ is called regular if every point of M lies in the domain of a regular chart. The following result shows that for a regular local transformation group Φ , Δ_Φ is regular in this sense.

3.1.17 Lemma. *Let $\Phi: G \times M \supseteq \mathcal{U} \rightarrow M$ be a regular local transformation group. Then each point of M lies in the domain of a regular chart for Δ_Φ , i.e., Δ_Φ is a regular distribution.*

Proof. Let $m \in M$ and pick a cubical chart (U, φ) as in 2.1.1 centered at m , $\varphi(U) = [-c, c]^n$. Since G acts regularly on M , there exists some neighborhood $V \subseteq U$ of m such that the intersection of each orbit S of G (i.e., each maximal integral manifold of Δ_Φ) with V is connected in S . This intersection is therefore a connected integral manifold of Δ_Φ , hence by 2.1.1 it is contained in a single slice U_a of φ . Now pick $c' \in (0, c)$ so that $U' := \varphi^{-1}((-c', c')^n) \subseteq V$. If S is any orbit of G that intersects U' then it also intersects V , hence $S \cap V \subseteq U_a$ for some a . Therefore, $S \cap U' \subseteq U'_a$. But since S is a maximal integral manifold we also have $U'_a \subseteq S$, so altogether we obtain $S \cap U' = U'_a$. \square

3.1.18 Example. Returning once more to the example from 3.1.4, let

$$\begin{aligned} U_+ &:= \{(x, y) \in \mathbb{R}^2 \mid x > 0\} \\ U_- &:= \{(x, y) \in \mathbb{R}^2 \mid x < 0\} \end{aligned}$$

and set

$$\begin{aligned} \varphi_\pm : U_\pm &\rightarrow \mathbb{R} \times \mathbb{R}_\pm \\ (x, y) &\mapsto \left(x, \frac{y}{x}\right) \end{aligned}$$

We show that (U_+, φ_+) is regular chart for the transformation group Φ from 3.1.4 (and analogously for (U_-, φ_-)). Note that $\varphi^{-1}(r, s) = (r, rs)$. From 3.1.10 we know that Δ_Φ is spanned by the vector field $X = x^2\partial_x + xy\partial_y$. In the chart (U_+, φ_+) , with coordinates $(r, s) := \varphi_+(x, y)$ we have

$$\varphi_*(\partial_x) = \frac{\partial r}{\partial x}\partial_r + \frac{\partial s}{\partial x}\partial_s = \partial_r - \frac{s}{r}\partial_s, \quad \varphi_*(\partial_y) = \frac{\partial r}{\partial y}\partial_r + \frac{\partial s}{\partial y}\partial_s = \frac{1}{r}\partial_s.$$

Therefore, X has the representation

$$\begin{aligned} \varphi_*X &= (\varphi^{-1})_1^2\varphi_*(\partial_x) + (\varphi^{-1})_1(\varphi^{-1})_2\varphi_*(\partial_y) \\ &= r^2\varphi_*(\partial_x) + rs\varphi_*(\partial_y) = r^2\partial_r. \end{aligned}$$

By 3.1.4 the non-trivial orbits, i.e., the maximal integral manifolds of Φ are half-rays emanating from $(0, 0)$, except for those lying on the y -axis. Hence each such orbit S is either contained in U_+ or in U_- . If $S \subseteq U_+$ then

$$S = \{(x, y) \in \mathbb{R}^2 \mid s = \frac{y}{x} = c \text{ and } x > 0\},$$

so S is a slice of φ_+ . This shows that (U_\pm, φ_\pm) are regular charts for Δ_Φ . Nevertheless, Φ is not regular because the points on the y -axis are trivial orbits, i.e., not every orbit of Φ has the same dimension. Removing the y -axis from $M = \mathbb{R}^2$ we obtain a regular local transformation group.

Finally, we have the following fundamental result on the space of orbits of a regular local transformation group:

3.1.19 Theorem. *Let $\Phi: G \times M \supseteq \mathcal{U} \rightarrow M$ be a regular local transformation group on an n -dimensional manifold M with k -dimensional orbits. Then the set M/G of orbits of Φ can be endowed with the structure of an $(n - k)$ -dimensional manifold with the following properties:*

- (i) The quotient map $\pi : M \rightarrow M/G$, $m \mapsto S_{\Phi, m}$ is a surjective submersion.
- (ii) m and m' belong to the same orbit if and only if $\pi(m) = \pi(m')$.
- (iii) For any $m \in M$, $\Delta_{\Phi}(m) = \ker T_m \pi$.

Proof. Surjectivity of π and (ii) are immediate from the definition. Since the orbits of Φ are precisely the leaves of the foliation induced by Δ_{Φ} , which is regular by 3.1.17, (i) is [11, 17.29]. For (iii), note that since π is constant on any orbit S of Φ , $T_m \Phi$ must vanish on the tangent space of S at $m \in S$. But S is an integral manifold of Δ_{Φ} , so $T_m S = \Delta_{\Phi}(m)$. Thus $\Delta_{\Phi}(m) \subseteq \ker(T_m \pi)$. Finally, since π is a submersion, $\dim \ker(T_m \pi) = k = \dim(\Delta_{\Phi}(m))$, so we have equality. \square

3.1.20 Remark. For later use we recall the construction of [11, 17.29]: Given a regular chart $(U, \varphi = (x^1, \dots, x^n))$, let $U' := \pi(U)$. Then the map $\varphi' : U' \rightarrow \mathbb{R}^{n-k}$, $m' \mapsto \text{pr}_2(\varphi(m))$, where m is any element of $\pi^{-1}(m') \cap U$ (recall that $\pi^{-1}(m') \cap U = U_a$ for some a and $\text{pr}_2 \circ \varphi|_{U_a} \equiv a$) is a typical chart for M/G . The local representation of π with respect to the standard charts then is

$$\begin{array}{ccc} \varphi' \circ \pi \circ \varphi^{-1} = \text{pr}_2 = (x^1, \dots, x^n) \mapsto (x^{k+1}, \dots, x^n) : & & \\ M \supseteq U & \xrightarrow{\pi} & U' \subseteq M/G \\ \varphi \downarrow & & \downarrow \varphi' \\ \varphi(U) & \xrightarrow{\text{pr}_2} & \varphi'(U) \end{array}$$

3.2 Symmetries of algebraic equations

From this point of the course onwards we will closely follow Olver's work [14]. Concerning notations, we will henceforth typically denote points in a smooth manifold M by x, y, \dots , since the manifolds we are interested in will mainly be subsets of spaces of independent and dependent variables of differential equations. As already announced before 1.1.13, we will from now on denote the group parameter with ε instead of t since we will often need t as a variable in a differential equation. Moreover, given a local Lie group action Φ on M we will often notationally suppress the Lie algebra homomorphism $v \mapsto \Phi(v)$ from (1.1.2), i.e., we will often simply write v instead of $\Phi(v) \in \mathfrak{X}_{\text{loc}}(M)$. Also, we will usually only write G instead of Φ .

By a system of algebraic equations we mean any system of equations

$$F_{\nu}(x) = 0, \quad \nu = 1, \dots, l, \quad (3.2.1)$$

where F_1, \dots, F_l are smooth real-valued functions on M . The term ‘algebraic’ is used to distinguish this situation from the case of differential equations to be considered later on. It does *not*, however, restrict the form of the F_{ν} (e.g., to polynomials). A *solution* of (3.2.1) is any point $x \in M$ such that $F_{\nu}(x) = 0$ for $\nu = 1, \dots, l$. A *symmetry group* of (3.2.1) is any local transformation group G on M that transforms any solution of (3.2.1) into another solution.

More generally, we define:

3.2.1 Definition. Let G be a local transformation group on a smooth manifold M . A subset $S \subseteq M$ is called *G-invariant*, and G is called a *symmetry group* of S if, whenever $x \in S$ and $g \in G$ is such that $g \cdot x$ is defined, then also $g \cdot x \in S$.

3.2.2 Example. (i) Let $M = \mathbb{R}^2$ and consider the one-parameter group of translations

$$G_c : (x, y) \mapsto (x + c\varepsilon, y + \varepsilon) \quad (c \in \mathbb{R}).$$

Then any line $\{x = cy + d\} \subseteq \mathbb{R}^2$ is an orbit of G_c , hence is invariant under G_c . Any G_c -invariant subset of M is a union of such orbits.

(ii) Again let $M = \mathbb{R}^2$ and for $\alpha \in \mathbb{R}$ let

$$G^\alpha : (x, y) \mapsto (\lambda x, \lambda^\alpha y) \quad (\lambda > 0).$$

Then $\{(0, 0)\}$, as well as the positive and negative x - and y -axes are G^α -invariant. Hence also the entire axes are invariant, being unions of invariant sets. Moreover, $\{xy = 0\}$ and $\{y = k|x|^\alpha\}$ (for $x < 0$ or $x > 0$) are invariant.

(iii) Most of the time we will be interested in invariant sets S that are subvarieties, given by the common zero set of smooth functions $F = (F_1, \dots, F_l)$:

$$S = S_F = \{x \mid F_\nu(x) = 0, \nu = 1, \dots, l\}.$$

(iv) If S_1, S_2 are invariant subsets, so are $S_1 \cup S_2$ and $S_1 \cap S_2$.

3.2.3 Definition. Let G be a local transformation group acting on M and let $F : M \rightarrow N$ be a map into a manifold N . F is called G -invariant if for all $x \in M$ and all $g \in G$ such that $g \cdot x$ is defined we have

$$F(g \cdot x) = F(x).$$

If $N = \mathbb{R}$ then F is simply called an invariant of G .

Obviously, $F = (F_1, \dots, F_l) : M \rightarrow \mathbb{R}^l$ is G -invariant if and only if each F_ν is an invariant of G .

3.2.4 Example. (i) For G_c the group of translations from 3.2.2 (i), the function

$$f(x, y) := x - cy$$

is an invariant: $f(x + c\varepsilon, y + \varepsilon) = f(x, y)$. Moreover, any invariant of G_c must be of the form $g(x - cy)$ for some smooth g .

(ii) As in 3.2.2 (ii), let $G^1 : (x, y) \mapsto (\lambda x, \lambda y)$ ($\lambda > 0$). Then $f(x, y) := x/y$ is an invariant for G^1 , defined on $\{y \neq 0\}$. Another invariant is $(x, y) \mapsto xy/(x^2 + y^2)$, defined on $\mathbb{R}^2 \setminus \{(0, 0)\}$. There is no smooth nonconstant invariant defined on all of \mathbb{R}^2 .

3.2.5 Remark. If $F : M \rightarrow \mathbb{R}^l$ is a G -invariant function then every level set of F is a G -invariant subset of M : if $F(x) = c$ and $g \cdot x$ is defined then also $F(g \cdot x) = F(x) = c$.

However, if the zero-set $\{F(x) = 0\}$ of a smooth map F is G -invariant then F itself need not be invariant. For example, $\{(x, y) \mid xy = 0\}$ is invariant under G^1 from 3.2.2 (ii), but $F(x, y) = xy$ is not G -invariant since $F(\lambda x, \lambda y) = \lambda^2 xy \neq F(x, y)$. To obtain a true statement we need to take all level sets into consideration:

3.2.6 Lemma. Let G be a local transformation group acting on M and let $F \in C^\infty(M, \mathbb{R}^l)$. The following are equivalent:

(i) F is invariant under G .

(ii) Every level set $\{F(x) = c\}$ ($c \in \mathbb{R}^l$) of F is invariant under G .

Proof. (i) \Rightarrow (ii): See 3.2.5.

(ii) \Rightarrow (i): Let x, g be such that $g \cdot x$ is defined and set $c := F(x)$. Then also $gx \in \{y \mid F(y) = c\}$, so $F(gx) = F(x)$. \square

The following result gives a simple, linear, criterion for a function to be an invariant of a group action. It is a first typical example of Lie theoretic methods in symmetry analysis. Recall from 3.1.5 that we always assume transformation groups to be connected.

3.2.7 Theorem. Let G be a local transformation group acting on M and let $f \in C^\infty(M, \mathbb{R})$. The following are equivalent:

(i) f is an invariant of G .

(ii) For every infinitesimal generator v of G we have $v(f) = 0$.

Proof. (i) \Rightarrow (ii): For clarity, we de-identify again and write $\Phi(g, x) = gx$ for the action of G . Let $v \in \mathfrak{g}$. Then since f is an invariant, given any $x \in M$, by 3.1.9 we obtain for ε small:

$$f(x) = f(\exp(\varepsilon v) \cdot x) \equiv f(\Phi(\exp(\varepsilon v), x)) = f(\text{Fl}_\varepsilon^{\Phi(v)}(x)) \equiv f(\text{Fl}_\varepsilon^v(x)).$$

Differentiating this expression with respect to ε at $\varepsilon = 0$ we get

$$v(f) \equiv \Phi(v)(f) = 0.$$

(ii) \Rightarrow (i): Since $v(f)$ vanishes identically on M we have

$$\frac{d}{d\varepsilon} f(\exp(\varepsilon v) \cdot x) = \frac{d}{d\varepsilon} f(\text{Fl}_\varepsilon^v(x)) = v(f)(\text{Fl}_\varepsilon^v(x)) = 0,$$

wherever defined, so $f(\exp(\varepsilon v) \cdot x) = f(x)$ wherever defined. Since G is connected, by 3.1.12 any $g \in \mathcal{U}_x$ can be written as a product of certain $\exp(\varepsilon_i v_i)$ ($v_i \in \mathfrak{g}$) satisfying (i)–(iii) from that result. Then 3.1.13 shows that we can iterate the above argument to conclude that $f(g \cdot x) = f(x)$ for all $g \in \mathcal{U}_x$. \square

It follows that, if $\{v_1, \dots, v_r\}$ is a basis of the local Lie algebra of infinitesimal generators of G (i.e., if $\{\Phi(v_1), \dots, \Phi(v_r)\}$ is a local basis of Δ_Φ) then f is an invariant of G (on the open set where $\{v_1, \dots, v_r\}$ is a basis) if and only if $v_k(f) = 0$ for $k = 1, \dots, r$. If G acts effectively, then by 1.1.17 we may take for $\{v_1, \dots, v_r\}$ any basis of \mathfrak{g} . Writing $v_k = \sum_{i=1}^n \xi_k^i \partial_{x_i}$ in local coordinates, this means that f has to satisfy the homogeneous system of linear PDEs of first order

$$v_k(f)(x) = \sum_{i=1}^n \xi_k^i(x) \frac{\partial f}{\partial x_i} = 0 \quad (k = 1, \dots, r). \quad (3.2.2)$$

3.2.8 Example. We return to the translation group G_c from 3.2.2 (i). Its infinitesimal generator is

$$v = \left. \frac{d}{d\varepsilon} \right|_0 (x + c\varepsilon, y + \varepsilon) = (c, 1) \equiv c\partial_x + \partial_y.$$

We already know that $f(x, y) = x - cy$ is an invariant, and indeed $v(f) = 0$.

In the case of the scale group G^α from 3.2.2 (ii) we have an action of the multiplicative group (\mathbb{R}^+, \cdot) on \mathbb{R}^2 whose generator therefore is $v = \partial_\lambda|_{\lambda=1}(\lambda x, \lambda^\alpha y) = x\partial_x + \alpha y\partial_y$ and it is easily checked that the infinitesimal criterion is satisfied for the invariants of G^α given in 3.2.4 (ii).

Turning now to symmetries of systems of algebraic equations, we begin by deriving a general criterion for the local invariance of submanifolds under local group actions.

3.2.9 Definition. Let G be a local transformation group acting on M . A subset $S \subseteq M$ is called *locally G -invariant* if for every $x \in S$ there exists a neighborhood $\tilde{\mathcal{U}}_x \subseteq \mathcal{U}_x$ (cf. 3.1.1) of the identity in G such that $g \cdot x \in S$ for all $g \in \tilde{\mathcal{U}}_x$. A smooth map $F : U \rightarrow N$ (U open in M) is called *locally G -invariant* if for each $x \in U$ there exists some neighborhood $\tilde{\mathcal{U}}_x \subseteq \mathcal{U}_x$ of e in G such that $F(g \cdot x) = F(x)$ for all $g \in \tilde{\mathcal{U}}_x$ with $g \cdot x \in U$. F is called *globally G -invariant* if $F(g \cdot x) = F(x)$ for all $x \in U$ and $g \in G$ such that $g \cdot x \in U$.

3.2.10 Example. Let G be the group of translations $(x, y) \mapsto (x + \varepsilon, y)$ on $M = \mathbb{R}^2$. Then the set $S := \{(x, 0) \mid -1 < x < 1\}$ is locally G -invariant, but not G -invariant.

Let

$$f(x, y) := \begin{cases} 0 & y \leq 0 \text{ or } y > 0 \text{ and } x > 0 \\ e^{-1/y} & y > 0 \text{ and } x < 0. \end{cases}$$

Then f is smooth and locally G -invariant on $U := \mathbb{R}^2 \setminus \{(0, y) \mid y \geq 0\}$: in fact, $f(x + \varepsilon, y) = f(x, y)$ for $|\varepsilon| < |x|$. But f is not globally G -invariant.

3.2.11 Theorem. Let N be an initial submanifold of M . The following are equivalent:

(i) N is locally G -invariant.

(ii) The infinitesimal generators of G are everywhere tangent to N , i.e.:

$$\forall x \in N \quad \forall v \in \mathfrak{g} : v|_x \in T_x N$$

Proof. If $\Phi : G \times M \supseteq \mathcal{U} \rightarrow M$ is the action of G then the above condition, written out, means (cf. (1.1.2)):

$$\forall x \in N \quad \forall v \in \mathfrak{g} : \Phi(v)|_x \in T_x N$$

(i) \Rightarrow (ii): Let $v \in \mathfrak{g}$, $x \in N$. Since N is locally G -invariant, for $|\varepsilon|$ small we have $\Phi(\exp(\varepsilon v), x) \in N$. As N is initial, $\varepsilon \mapsto \Phi(\exp(\varepsilon v), x)$ is smooth as a map into N as well. The derivative of this curve at $\varepsilon = 0$ therefore is an element of $T_x N$. This implies (ii) since by 3.1.9 we have

$$\Phi(v)(x) = \left. \frac{d}{d\varepsilon} \right|_0 \Phi(\exp(\varepsilon v), x).$$

(ii) \Rightarrow (i): By assumption, any $\Phi(v)$ is tangent to N , hence by [11, 17.16] it can be viewed as a vector field on N , and so the restriction of its flow to N is a local diffeomorphism on N (cf. [11, 7.2]). Given $x \in N$, $\text{Fl}_\varepsilon^{\Phi(v)}(x)$ exists for $|\varepsilon|$ small. Using 3.1.9 we therefore have for $|\varepsilon|$ small

$$\exp(\varepsilon v) \cdot x = \Phi(\exp(\varepsilon v), x) = \text{Fl}_\varepsilon^{\Phi(v)}(x) \in N. \quad (3.2.3)$$

Next, choose a basis v_1, \dots, v_k of \mathfrak{g} . Then as in the last part of the proof of 2.2.3 it follows that there exists some $c > 0$ such that $\text{Fl}_{\varepsilon_1}^{\Phi(v_1)} \circ \dots \circ \text{Fl}_{\varepsilon_k}^{\Phi(v_k)}(x)$ exists for all $|\varepsilon_i| < c$. In addition, we may assume c so small that $\exp(\varepsilon_1 v_1) \cdots \exp(\varepsilon_k v_k) \in \mathcal{U}_x$ for $|\varepsilon_i| < c$. Then (i)–(iii) of 3.1.12 hold and so, by 3.1.13, iterating (3.2.3) gives

$$\exp(\varepsilon_1 v_1) \cdots \exp(\varepsilon_k v_k) \cdot x = \text{Fl}_{\varepsilon_1}^{\Phi(v_1)} \circ \dots \circ \text{Fl}_{\varepsilon_k}^{\Phi(v_k)}(x) \in N$$

for these ε_i . Therefore, defining the neighborhood $\tilde{\mathcal{U}}_x := \{\exp(\varepsilon_1 v_1) \dots \exp(\varepsilon_k v_k) \mid |\varepsilon_i| < c\}$ of $e \in G$ (shrinking c if necessary, cf. [11, 8.5]) we obtain that $g \cdot x \in N$ for all $g \in \tilde{\mathcal{U}}_x$. \square

3.2.12 Remark. As the proof of 3.2.11 shows, the implication (ii) \Rightarrow (i) remains correct even for N an immersive submanifold of M . However, (i) \Rightarrow (ii) is not true in this generality: To see this, equip $M = \mathbb{R}^2$ with a new manifold structure N whose charts are $\varphi_a : (x, a) \mapsto x$ ($a \in \mathbb{R}$), see [11, Ex. 14.3]. Then N is a one-dimensional immersive submanifold of M with underlying set \mathbb{R}^2 , hence is (even globally) invariant under *any* local group action on M . But clearly not every group action has generators tangential to N (i.e., horizontal).

As the most important special case of 3.2.11 we consider zero sets of smooth maps:

3.2.13 Corollary. *Let G be a local transformation group acting on M , $\dim(M) = m$. Let $F : M \rightarrow \mathbb{R}^l$ ($l \leq m$) be smooth and of maximal rank ($= l$) at every solution x of the system $F_\nu(x) = 0$ ($1 \leq \nu \leq l$). Then G is a symmetry group of this system if and only if*

$$\forall v \in \mathfrak{g} \ \forall \nu = 1, \dots, l \ \forall x \in M \text{ with } F(x) = 0 : v(F_\nu)|_x = 0 \quad (3.2.4)$$

Proof. By [10, 1.1.23], $N := \{x \in M \mid F(x) = 0\}$ is a regular (hence in particular initial) submanifold of M . Thus by 3.2.11, N is locally invariant under G if and only if $v|_x \in T_x N$ for all $x \in N$ and $v \in \mathfrak{g}$. By [10, 1.1.25],

$$T_x N = \ker(T_x F) = \bigcap_{\nu=1}^l \ker(T_x F_\nu)$$

for all $x \in N$. This implies that N is locally G -invariant if and only if for each $x \in N$, i.e., for each $x \in M$ with $F(x) = 0$, each $v \in \mathfrak{g}$, and each $\nu = 1, \dots, l$ we have that

$$0 = T_x F_\nu(v|_x) = v(F_\nu)|_x.$$

It remains to show that local invariance implies invariance of N under G . Thus let $x \in N$ and $g \in \mathcal{U}_x$. By 3.1.12, $g = \exp(t_1 v_1) \dots \exp(t_n v_n)$ with properties (i)–(iii) from that result. To show that $g \cdot x \in N$, we proceed by induction. Suppose we already know that $g_i \cdot x \in N$, where $g_i := \exp(t_{i+1} v_{i+1}) \dots \exp(t_n v_n)$ and set $A_i := \{t \in [0, t_i] \mid (\exp(t v_i) \cdot g_i) \cdot x \in N\}$. Then $0 \in A_i$ and A_i is closed since N is. If $t \in A_i$ then by local invariance of N there exists some $\alpha > 0$ such that for all $|s| < \alpha$ we have $\exp(s v_i) \cdot [(\exp(t v_i) \cdot g_i) \cdot x] \in N$. Here, $h_i := \exp(t v_i) \cdot g_i \in \mathcal{U}_x$ and $\exp(s v_i) \in \mathcal{U}_{h_i \cdot x}$. Also, for $|s| < \alpha$ and $t + s \in [0, t_i]$, $\exp(s v_i) \cdot \exp(t v_i) \cdot g_i = \exp((s + t) v_i) \cdot g_i \in \mathcal{U}_x$ by 3.1.12 (iii). Hence 3.1.1 (i) gives $[(\exp((s + t) v_i) \cdot g_i)] \cdot x = [\exp(s v_i) \cdot (\exp(t v_i) \cdot g_i)] \cdot x = \exp(s v_i) \cdot [(\exp(t v_i) \cdot g_i) \cdot x] \in N$ for these s , which establishes that A_i is also open in $[0, t_i]$. Altogether, $A_i = [0, t_i]$, and our claim follows. \square

3.2.14 Example. (i) Let $G = SO(2)$ be the rotation group acting on $M = \mathbb{R}^2$, with infinitesimal generator

$$v = \left. \frac{d}{d\varepsilon} \right|_0 (x \cos \varepsilon - y \sin \varepsilon, x \sin \varepsilon + y \cos \varepsilon) = -y \partial_x + x \partial_y$$

Then $S^1 = \{x^2 + y^2 = 1\}$ is clearly invariant under G . We can also see this using 3.2.13: let $F(x, y) := x^2 + y^2 - 1$. Then F is of maximal rank ($= 1$) on its zero set S^1 and $v(F) = -2xy + 2xy = 0$.

(ii) Let $H(x, y) = y^2 - 2y + 1$. Then the zero set of H is the horizontal line $\{y = 1\}$, which is manifestly not invariant under G . Nevertheless, $v(H) = 2xy - 2x = 2x(y - 1)$ vanishes on this zero set. However, also $TH = (0, 2y - 2)$ vanishes there, so 3.2.13 does not apply. This demonstrates that the maximal rank condition in 3.2.13 cannot be dropped.

The following criterion will turn out to be useful many times later on:

3.2.15 Theorem. (*Hadamard's Lemma*) Let $F : M^m \rightarrow \mathbb{R}^l$ ($l \leq m$) be of maximal rank on the subvariety $S_F := \{x \in M \mid F(x) = 0\}$. Then a smooth real-valued function $f : M \rightarrow \mathbb{R}$ vanishes on S_F if and only if there exist $Q_1, \dots, Q_l \in C^\infty(M, \mathbb{R})$ such that

$$\forall x \in M : f(x) = Q_1(x)F_1(x) + \dots + Q_l(x)F_l(x).$$

Proof. The condition is clearly sufficient. Conversely, suppose that f vanishes on S_F and let $x \in S_F$. Since F is a submersion, by [10, 1.1.7] there exists a chart $(\varphi = (y^1, \dots, y^m), U)$ centered at x such that $F_\varphi(y) := F \circ \varphi^{-1}(y^1, \dots, y^m) = (y^1, \dots, y^l)$. It follows that $f_\varphi := f \circ \varphi^{-1}$ vanishes on $\varphi(U) \cap (\{0\} \times \mathbb{R}^{m-l})$. We can suppose that $\varphi(U)$ is a ball around 0 in \mathbb{R}^m . Then for $y = (y', y'') := (y^1, \dots, y^l, y^{l+1}, \dots, y^m)$ we have

$$\begin{aligned} f_\varphi(y', y'') &= f_\varphi(y', y'') - f_\varphi(0, y'') = \int_0^1 \partial_t f_\varphi(ty', y'') dt \\ &= \sum_{i=1}^l \int_0^1 \frac{\partial f_\varphi}{\partial y_i}(ty', y'') \cdot y^i dt =: \sum_{i=1}^l Q_\varphi^i(y) \cdot y^i = \sum_{i=1}^l Q_\varphi^i(y) \cdot (F_i)_\varphi(y). \end{aligned}$$

Therefore, setting $Q_U^i := Q_\varphi^i \circ \varphi \in C^\infty(U)$ we obtain $f|_U = \sum_{i=1}^l Q_U^i \cdot F_i|_U$.

On the other hand, if $x \notin S_F$ then there exists some open neighborhood U of x and some $i \in \{1, \dots, l\}$ such that $F_i(y) \neq 0$ for all $y \in U$. Then we set $Q_U^i := f/F_i$ and $Q_U^j := 0$ for $j \neq i$ to again obtain $f|_U = \sum_{i=1}^l Q_U^i \cdot F_i|_U$. Now pick a locally finite open cover $(U_\alpha)_{\alpha \in A}$ of neighborhoods as above and a subordinate partition of unity $(\chi_\alpha)_{\alpha \in A}$ with $\text{supp}(\chi_\alpha) \subseteq U_\alpha$ for all α . Then

$$f = \sum_{\alpha} \chi_\alpha f = \sum_{\alpha} \chi_\alpha \sum_{i=1}^l Q_{U_\alpha}^i F_i =: \sum_{i=1}^l Q_i F_i.$$

□

It follows that an equivalent reformulation of (3.2.4) is that for any $v \in \mathfrak{g}$ there exist smooth functions $Q_{\nu\mu} : M \rightarrow \mathbb{R}$ ($\mu, \nu = 1, \dots, l$) such that

$$v(F_\nu)(x) = \sum_{\mu=1}^l Q_{\nu\mu}(x) F_\mu(x) \quad \nu = 1, \dots, l, \quad x \in M. \quad (3.2.5)$$

3.2.16 Example. The functions Q_ν in 3.2.15 are in general not unique: Let $M = \mathbb{R}^3$ and $F(x, y, z) := (x, y)$. Then the function $f(x, y, z) = xz + y^2$ vanishes on S_F (which is the z -axis) and we have

$$f = zF_1 + yF_2 = (z - y)F_1 + (x + y)F_2.$$

If $f = \sum_{\nu} Q_{\nu} F_{\nu} = \sum_{\nu} \tilde{Q}_{\nu} F_{\nu}$, then the differences $R_{\nu} := Q_{\nu} - \tilde{Q}_{\nu}$ satisfy the homogeneous system

$$\sum_{\nu=1}^l R_{\nu}(x) F_{\nu}(x) = 0. \quad (3.2.6)$$

For such functions we have:

3.2.17 Proposition. *Let $F : M^m \rightarrow \mathbb{R}^l$ be of maximal rank on $S_F = \{x \in M \mid F(x) = 0\}$. If $R_1, \dots, R_l \in C^{\infty}(M, \mathbb{R})$ satisfy (3.2.6), then each R_{ν} vanishes identically on S_F . Equivalently, there exist $S_{\nu}^{\mu} \in C^{\infty}(M, \mathbb{R})$ ($\nu, \mu = 1, \dots, l$) such that, for all $x \in M$ and all $\nu = 1, \dots, l$*

$$R_{\nu} = \sum_{\mu=1}^l S_{\nu}^{\mu} F_{\mu}. \quad (3.2.7)$$

In addition, the S_{ν}^{μ} can be chosen to be anti-symmetric: $S_{\nu}^{\mu} = -S_{\mu}^{\nu}$. In this case, (3.2.7) is necessary and sufficient for (3.2.6).

Proof. As in the proof of 3.2.15, by using a partition of unity and suitable charts we may reduce the proof to neighborhoods of points with either $x \in S_F$ or $x \notin S_F$.

1) If $x \in S_F$, using a chart as in 3.2.15, we may suppose that M is a ball in \mathbb{R}^m and $F = (x^1, \dots, x^m) \mapsto (x^1, \dots, x^l)$. Then (3.2.6) reads

$$\sum_{\nu=1}^l R_{\nu}(x) x^{\nu} = 0. \quad (3.2.8)$$

We show by induction that (3.2.8) implies the existence of a skew-symmetric matrix S_{ν}^{μ} of smooth functions satisfying (3.2.7).

For $l = 1$, (3.2.8) reduces to $R_1(x) x^1 = 0$ for all x , so by continuity $R_1(x) = 0$ for all x , and we can choose $S_1^1 = 0$.

$l - 1 \rightarrow l$: Setting $x' := (x^{l+1}, \dots, x^m)$, (3.2.8) implies that $R_l(0, \dots, 0, x^l, x') \cdot x^l = 0$, so in fact $R_l(0, \dots, 0, x^l, x') = 0$ for all (x^l, x') . Applying 3.2.15 to $F := (x^1, \dots, x^m) \mapsto (x^1, \dots, x^{l-1})$, it follows that there exist smooth functions S_l^{μ} ($\mu = 1, \dots, l - 1$) of x with $R_l(x) = \sum_{\mu=1}^{l-1} S_l^{\mu}(x) x^{\mu}$. By (3.2.8), therefore,

$$0 = \sum_{\mu=1}^{l-1} (R_{\mu}(x) x^{\mu} + S_l^{\mu}(x) x^{\mu} x^l) = \sum_{\mu=1}^{l-1} (R_{\mu}(x) + S_l^{\mu}(x) x^l) x^{\mu} =: \sum_{\mu=1}^{l-1} \tilde{R}_{\mu} x^{\mu}.$$

By our induction hypothesis there exist smooth functions \tilde{S}_{ν}^{μ} ($1 \leq \mu, \nu \leq l - 1$) such that $\tilde{R}_{\nu} = \sum_{\mu=1}^{l-1} \tilde{S}_{\nu}^{\mu} x^{\mu}$ for $1 \leq \nu \leq l - 1$ and such that \tilde{S}_{ν}^{μ} is skew-symmetric. By definition, $R_{\mu} = \tilde{R}_{\mu} - S_l^{\mu} x^l$ for $1 \leq \mu \leq l - 1$. Therefore,

$$\begin{pmatrix} R_1 \\ \vdots \\ R_{l-1} \\ R_l \end{pmatrix} = \begin{pmatrix} \tilde{S}_1^1 & \dots & \tilde{S}_1^{l-1} & -S_1^l \\ \vdots & & \vdots & \vdots \\ \tilde{S}_{l-1}^1 & \dots & \tilde{S}_{l-1}^{l-1} & -S_{l-1}^l \\ S_1^1 & \dots & S_l^{l-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} x^1 \\ \vdots \\ x^{l-1} \\ x^l \end{pmatrix} =: S \cdot x,$$

giving the desired skew-symmetric $l \times l$ matrix S .

2) Now suppose that $x \notin S_F$. Again we proceed by induction. For $l = 1$, $F_1(x) \neq 0$ and $R_1 \cdot F_1 = 0$ imply that we can write $R_1(x) = 0 = 0 \cdot F_1 =: S_1^1 \cdot F_1$ on a neighborhood of x where $F_1 \neq 0$.

Next, assume that the result is already proved for $l - 1$. Since $F(x) \neq 0$, some component of $F(x)$ must be non-zero, and without loss of generality we may suppose that $F_{l-1}(x) \neq 0$ (the proof below works the same way for any other component as well). Then on a neighborhood U of x where $F_{l-1}(x) \neq 0$ we have

$$R_1 F_1 + \cdots + R_{l-2} F_{l-2} + \left(R_{l-1} + R_l \frac{F_l}{F_{l-1}} \right) F_{l-1} = 0,$$

and so by our induction assumption (and shrinking U if necessary) there exists a skew-symmetric $(l - 1) \times (l - 1)$ matrix \hat{S}_ν^μ such that

$$\begin{pmatrix} R_1 \\ \vdots \\ R_{l-2} \\ R_{l-1} + R_l \frac{F_l}{F_{l-1}} \end{pmatrix} = \begin{pmatrix} \hat{S}_1^1 & \cdots & \hat{S}_1^{l-1} \\ \vdots & & \vdots \\ \hat{S}_{l-2}^1 & \cdots & \hat{S}_{l-2}^{l-1} \\ \hat{S}_{l-1}^1 & \cdots & \hat{S}_{l-1}^{l-1} \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ \vdots \\ F_{l-2} \\ F_l \end{pmatrix}.$$

Consequently,

$$\begin{pmatrix} R_1 \\ \vdots \\ R_{l-2} \\ R_{l-1} \\ R_l \end{pmatrix} = \begin{pmatrix} \hat{S}_1^1 & \cdots & \hat{S}_1^{l-1} & 0 \\ \vdots & & \vdots & \vdots \\ \hat{S}_{l-2}^1 & \cdots & \hat{S}_{l-2}^{l-1} & 0 \\ \hat{S}_{l-1}^1 & \cdots & \hat{S}_{l-1}^{l-1} & -R_l/F_{l-1} \\ 0 & \cdots & 0 & R_l/F_{l-1} \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ \vdots \\ F_{l-2} \\ F_{l-1} \\ F_l \end{pmatrix},$$

which gives the claim also in this case.

Conversely, if (3.2.7) is satisfied, then denoting by $\langle \cdot, \cdot \rangle$ the standard scalar product on \mathbb{R}^l we have

$$\langle R(x), F(x) \rangle = \langle S(x)F(x), F(x) \rangle = 0$$

since $S^\top = -S$. This is (3.2.6). □

3.3 Invariance and functional dependence

Our goal in this section is to determine ‘how many’ invariants a given group action possesses. We first observe that if ζ^1, \dots, ζ^k are invariants (either local or global) of a group G and $F = F(\zeta^1, \dots, \zeta^k)$ is any smooth function then also

$$\zeta(x) := F(\zeta^1(x), \dots, \zeta^k(x))$$

is an invariant. However, ζ is completely determined by ζ^1, \dots, ζ^k and so adds no additional information. We therefore want to determine invariants up to this kind of relation.

3.3.1 Definition. Let $\zeta^1, \dots, \zeta^k \in C^\infty(M, \mathbb{R})$. Then

- (i) ζ^1, \dots, ζ^k are called *functionally dependent* if for each $x \in M$ there exists a neighborhood U of x and a smooth map $F : \mathbb{R}^k \rightarrow \mathbb{R}$ that does not vanish identically on any open subset of \mathbb{R}^k such that

$$F(\zeta^1(x), \dots, \zeta^k(x)) = 0 \tag{3.3.1}$$

for all $x \in U$.

(ii) ζ^1, \dots, ζ^k are called *functionally independent* if they are not functionally dependent when restricted to any open subset U of M . Equivalently, if F as in (i) is such that (3.3.1) holds for all x in some open subset U of M then $F(z_1, \dots, z_k) = 0$ for all z in some open subset of \mathbb{R}^k (which contains $(\zeta^1, \dots, \zeta^k)(U')$ for some $U' \subseteq U$).

3.3.2 Example. (i) The functions x/y and $xy/(x^2+y^2)$ are functionally dependent on $\{(x, y) \mid y \neq 0\}$ because

$$\frac{xy}{x^2+y^2} = \frac{x/y}{1+(x/y)^2} = f(x/y).$$

On the other hand, x/y and $x+y$ are functionally independent where defined since if $F(x+y, x/y) \equiv 0$ for (x, y) in any open subset of \mathbb{R}^2 then since $(x, y) \mapsto (x+y, x/y)$ is a local diffeomorphism, F has to vanish identically on some open subset of the image of this set.

(ii) The functions

$$\eta(x, y) = x, \quad \zeta(x, y) = \begin{cases} x & y \leq 0 \\ x + e^{-1/y} & y > 0 \end{cases}$$

are functionally dependent on $\{y < 0\}$, independent on $\{y > 0\}$, but *neither* on the entire space \mathbb{R}^2 .

For a characterization of functional (in)dependence we need two auxiliary results, both of which are of independent interest as well.

3.3.3 Proposition. *Let M be a second countable smooth manifold and let S be a closed subset of M . Then there exists a smooth function $f: M \rightarrow \mathbb{R}$ such that $S = \{x \in M \mid f(x) = 0\}$.*

Proof. Since $M \setminus S$ is open and M is second countable, $M \setminus S$ can be written as a locally finite union of countably many sets V_j ($j \in \mathbb{N}$) and there exist functions $\chi_j \in \mathcal{C}^\infty(M)$ with $\chi_j \geq 0$ and $\chi_j > 0$ precisely on V_j (see the proof of [9, 2.3.10]). Then set

$$f := \sum_{j \in \mathbb{N}} \chi_j.$$

Since $(V_j)_j$ is locally finite, locally around any point in M only finitely many summands are nonzero, so $f \in \mathcal{C}^\infty(M)$. Moreover, for any $x \in M$ we have

$$f(x) = 0 \Leftrightarrow \forall j : \chi_j(x) = 0 \Leftrightarrow x \in \bigcap_{j \in \mathbb{N}} (M \setminus V_j) = M \setminus \bigcup_{j \in \mathbb{N}} V_j = S.$$

□

In the next theorem, by a *critical point* of a smooth map $f: M \rightarrow N$ we mean a point $x \in M$ where $T_x f$ is not surjective. A point $y \in N$ is called a *critical value* of f if there exists a critical point $x \in M$ of f such that $y = f(x)$.

3.3.4 Theorem. (Sard's Theorem) *Let $f: M \rightarrow N$ be a smooth map between smooth manifolds. Then the set of critical values of f in N has measure zero, in the sense that in any chart of N the image of the set of critical values in that chart domain has Lebesgue measure zero.*

A proof of this result would take us too far afield, so we refer to [8, 3.2].

Based on the previous results we now have:

3.3.5 Theorem. Let $\zeta = (\zeta^1, \dots, \zeta^k)$ be a smooth map from M to \mathbb{R}^k . Then the following are equivalent:

- (i) ζ^1, \dots, ζ^k are functionally dependent on M .
- (ii) $\forall x \in M : \text{rk}(T_x \zeta) < k$.

Proof. (i) \Rightarrow (ii): Suppose that for some $x \in M$ we had $\text{rk}(T_x \zeta) = k$. Since the rank can't fall locally, there would then exist some open set U around x with $\text{rk}(T_{x'} \zeta) = k$ for all $x' \in U$. We may assume U small enough that there is some F as in 3.3.1 with $F(\zeta^1(x'), \dots, \zeta^k(x')) \equiv 0$ on U . Since ζ is a submersion on U , it is an open map. Thus F vanishes on the open set $\zeta(U)$, a contradiction.

(ii) \Rightarrow (i): Let U be open and relatively compact in M . By 3.3.3 there exists some $F \in \mathcal{C}^\infty(\mathbb{R}^k)$ such that $F(z) = 0$ if and only if $z \in \zeta(\overline{U})$. Since $\zeta(\overline{U}) \subseteq \{\zeta(x) \mid \text{rk}(T_x \zeta) < k\}$ and the latter set has measure zero by Sard's theorem 3.3.4, $\zeta(\overline{U})$ does not contain any non-empty open set. Thus ζ^1, \dots, ζ^k are functionally dependent. \square

3.3.6 Theorem. Let G act semi-regularly on the m -dimensional manifold M with k -dimensional orbits and let $x_0 \in M$. Then there exists some open neighborhood U of x_0 such that there are precisely $m - k$ functionally independent local invariants $\zeta^1, \dots, \zeta^{m-k}$ of G on U . Any other invariant of G on U is of the form

$$f(x) = F(\zeta^1(x), \dots, \zeta^{m-k}(x))$$

for some smooth function F . If the action of G is regular then the invariants can be taken to be globally invariant on a neighborhood of x_0 .

Proof. By 3.1.16, any orbit of G is a leaf of the k -dimensional integrable distribution Δ_Φ . Thus the classical Frobenius theorem 2.1.1 yields a cubic chart $(\varphi = (x^1, \dots, x^m), U)$ centered at x_0 , $\varphi(U) = (-c, c)^m$, such that any orbit of G that intersects U does so in a (by 2.2.14 at most countable) union of slices $U_a = \varphi^{-1}(\mathbb{R}^k \times \{a\})$. For $1 \leq i \leq m - k$, let $\zeta^i : U \rightarrow \mathbb{R}$, $\zeta^i := x^{k+i} (= \varphi^{k+i})$. Then by definition, each ζ^i is constant on each slice, hence is locally constant on any orbit that intersects U , hence is a local invariant. Any other local invariant f must be constant on each slice. In terms of the chart φ this means that f does in fact not depend on the variables x^1, \dots, x^k , and therefore is a function of the remaining variables x^{k+i} ($1 \leq i \leq m - k$) only, i.e., of the ζ^i .

Finally, if the action of G is regular, then by 3.1.17 the chart (φ, U) can be chosen such that each G -orbit intersects U in at most one slice. Then the ζ^i are constant on this slice, i.e., on $U \cap S$, so they are global invariants. \square

Since $\zeta = (\zeta^1, \dots, \zeta^{m-k})$ has maximal rank, the ζ^i are functionally independent by 3.3.5. Such a family of invariants (i.e., a functionally independent family such that any other invariant is a function of the members of the family) is called a *complete set of functionally independent invariants*.

3.3.7 Proposition. Let G act semi-regularly on the m -dimensional manifold M with k -dimensional orbits. Then:

- (i) Any complete set of functionally independent local invariants of G has $m - k$ elements.
- (ii) If $\eta^1, \dots, \eta^{m-k}$ is a set of functionally independent local invariants, then locally around any point in its domain it is complete.

Proof. (i) Let $\eta = (\eta^1, \dots, \eta^l)$ and $(\theta^1, \dots, \theta^p)$ be two sets of functionally independent invariants of G and assume that $l > p$. Since θ is complete, there exists some smooth map $F : \Omega \rightarrow \mathbb{R}^l$ with $\Omega \subseteq \mathbb{R}^p$ open such that $\eta = F \circ \theta$. But then for any x , $\text{rk}(T_x \eta) \leq \text{rk} T_{\theta(x)}(F) \leq p < l$, a contradiction to 3.3.5. By symmetry, $l = p$, and by 3.3.6 it follows that in fact $l = m - k$.

(ii) Suppose that f is some invariant defined on the domain Ω of $\eta = (\eta^1, \dots, \eta^{m-k})$ and let $x \in \Omega$. Also, pick $\zeta = (\zeta^1, \dots, \zeta^{m-k})$ around x as in 3.3.6. Then there exists some $F : \mathbb{R}^{m-k} \rightarrow \mathbb{R}^{m-k}$, such that $\eta = F \circ \zeta$ near x . Since $\text{rk}(\eta) = \text{rk}(\zeta) = m - k$, we also must have $\text{rk}(F) = m - k$, i.e., F is a local diffeomorphism. Since f is a smooth function of ζ near x it is therefore also a smooth function of η . \square

3.3.8 Proposition. *Let G act semi-regularly on the m -dimensional manifold M with k -dimensional orbits and let $\zeta^1, \dots, \zeta^{m-k}$ be a complete set of functionally independent invariants defined on an open subset U of M . If a subvariety $S_F = \{x \in M \mid F(x) = 0\}$ (with $F \in C^\infty(M, \mathbb{R}^l)$) is G -invariant, then for each solution $x_0 \in S_F \cap U$ there is a neighborhood $\tilde{U} \subseteq U$ of x_0 and a smooth function f such that the solution set of the corresponding invariant $\tilde{F} = x \mapsto f(\zeta^1(x), \dots, \zeta^{m-k}(x))$ on \tilde{U} coincides with that of F , i.e.,*

$$S_F \cap \tilde{U} = S_{\tilde{F}} \cap \tilde{U} = \{x \in \tilde{U} \mid f(\zeta^1(x), \dots, \zeta^{m-k}(x)) = 0\}.$$

Proof. Since (by 3.3.5) the rank of $(\zeta^1, \dots, \zeta^{m-k})$ is $m - k$ on U we may find coordinates (y^1, \dots, y^m) on a neighborhood \tilde{U} of x_0 such that $y^i = \zeta^i$ for $1 \leq i \leq m - k$. In fact, the remaining k coordinates can be determined by picking suitable x^{i_j} (called *parametric variables*) from any given coordinates (x^1, \dots, x^m) . The change of coordinates is then of the form $y = \psi(x) = (\zeta(x), \hat{x})$, with \hat{x} the parametric variables.

We show that these coordinates are then flat for Δ_Φ : Each ζ^j is constant on the orbits of Δ_Φ , hence if $v_1, \dots, v_k \in \mathfrak{g}$ are such that $\Delta_\Phi(x) = \text{span}(v_1|_x, \dots, v_k|_x)$ then

$$T_x \zeta^j(v_i|_x) = v_i(\zeta^j)|_x = \left. \frac{d}{dt} \right|_0 \zeta^j(\text{Fl}_t^{v_i}(x)) = 0$$

for all $x \in \tilde{U}$. Therefore, $v_i|_x \in \bigcap_{j=1}^{m-k} \ker(T_x \zeta^j) = \ker T_x \zeta$, and since $\dim(\ker(T_x \zeta)) = k$, $\Delta_\Phi(x) = \text{span}(v_1|_x, \dots, v_k|_x) = \ker T_x \zeta$. On the other hand, $\ker(T_x \zeta)$ is spanned by $\partial_{y^{m-k+1}}, \dots, \partial_{y^m}$ because $T \zeta^i(\partial_{y^j}) = \frac{\partial y^i}{\partial y^j} = 0$ for $i \in \{1, \dots, m - k\}$ and $j \in \{m - k + 1, \dots, m\}$.

In terms of the new variables we can write $F(x) = F^*(y) = F^*(\zeta(x), \hat{x})$, where $F^* := F \circ \psi^{-1}$. Denoting by \hat{x}_0 the value of the parametric variables at x_0 we now set $f(z) := F^*(z, \hat{x}_0)$, and

$$\tilde{F}(x) := f(\zeta(x)) = f(\zeta^1(x), \dots, \zeta^{m-k}(x)).$$

By assumption, S_F is G -invariant. Also, the orbits of G intersect U in the slices $\{\zeta(x) = c\}$. Therefore, $F(x) = F^*(\zeta(x), \hat{x}) = 0$ if and only if $\tilde{F}(x) = F^*(\zeta(x), \hat{x}_0) = 0$ since $(\zeta(x), \hat{x})$ and $(\zeta(x), \hat{x}_0)$ lie in the same slice. \square

We now turn to the problem of actually calculating invariants of a local transformation group. To begin with, we consider the case of a one-parameter group G with infinitesimal generator $v \in \mathfrak{X}_{\text{loc}}(M)(M)$. In terms of local coordinates x^1, \dots, x^n we can write

$$v = \xi^1(x) \partial_{x^1} + \dots + \xi^m(x) \partial_{x^m}$$

for certain smooth functions ξ^i . Then by 3.2.7, a local invariant f of G is a solution of the linear homogeneous first order PDE

$$v(f) = \xi^1(x) \frac{\partial f}{\partial x^1} + \cdots + \xi^m(x) \frac{\partial f}{\partial x^m} = 0. \quad (3.3.2)$$

By 3.3.6 we know that if $v|_x \neq 0$ then locally around x there exist $m-1$ functionally independent invariants, i.e., $m-1$ functionally independent solutions of (3.3.2). In PDE theory, the method of choice for solving (3.3.2) is the *method of characteristics*, cf. [4, Sec. 3.2]. This basically consists in solving the corresponding system of ODEs given by

$$\frac{dx^i}{dt} = \xi^i(x(t)) \quad (1 \leq i \leq m). \quad (3.3.3)$$

Solutions of (3.3.2) are then functions that are constant along solutions of (3.3.3). Such functions are also called *first integrals* of (3.3.3). In this terminology, 3.3.6 says that (3.3.3) possesses $m-1$ functionally independent first integrals locally around any point.

3.3.9 Remark. Using the straightening-out theorem (cf. [11, 17.14]) it is very easy to see all of the above directly. In fact, in an appropriate coordinate system y^1, \dots, y^m we have $v = \partial_{y^1}$ near the point x . It is then immediate that the functions y^2, \dots, y^m form a complete set of functionally independent invariants of Fl^v . Furthermore, in these coordinates (3.3.3) reads $\frac{dy^1}{dt} = 1$, and $\frac{dy^i}{dt} = 0$ for $i = 2, \dots, m$, with general solution $y^1(t) = t + c_1$, $y^i = c_i$ for $i = 2, \dots, m$. A complete set of functionally independent first integrals then of course is also given by y^2, \dots, y^m , because these functions are constant along this general solution of the ODE.

3.3.10 Example. Consider the rotation group $SO(2)$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ with infinitesimal generator $v = -y\partial_x + x\partial_y$. Then the characteristic system reads

$$\frac{dx}{dt} = -y \quad \frac{dy}{dt} = x.$$

A first integral is $f(x, y) = x^2 + y^2$, since

$$\begin{aligned} \frac{d}{dt} f(x(t), y(t)) &= \partial_x f(x(t), y(t))(-y(t)) + \partial_y f(x(t), y(t))(x(t)) \\ &= -2x(t)y(t) + 2x(t)y(t) = 0. \end{aligned}$$

Any other first integral, i.e., any other invariant of v , is a function of f .

In the remainder of this section we want to explore how invariance of functions or subvarieties under a local transformation group can be expressed in terms of quotient manifolds, using 3.1.19. The main result is as follows:

3.3.11 Theorem. *Let G be a local transformation group on M that acts regularly on the m -dimensional manifold M with k -dimensional orbits.*

- (i) *A smooth map $F : M \rightarrow \mathbb{R}^l$ is G -invariant if and only if there exists a smooth function $\tilde{F} : M/G \rightarrow \mathbb{R}^l$ such that $F(x) = \tilde{F}(\pi(x))$ for all $x \in M$.*
- (ii) *Let $F \in C^\infty(M, \mathbb{R}^l)$. Then the corresponding subvariety $\mathcal{S}_F = \{x \in M \mid F(x) = 0\}$ is G -invariant if and only if there exists a smooth map $\tilde{F} : M/G \rightarrow \mathbb{R}^l$ such that, for all $x \in M$, $F(x) = 0$ if and only if $\tilde{F}(\pi(x)) = 0$.*
- (iii) *An n -dimensional regular submanifold N of M is G -invariant if and only if there exists a smooth $(n-k)$ -dimensional regular submanifold $\tilde{N} = N/G$ of M/G such that $N = \pi^{-1}(\tilde{N})$ (and therefore $\tilde{N} = \pi(N)$).*

Proof. (i) By definition, π is invariant under G , so any $\tilde{F} \circ \pi$ is G -invariant as well. Conversely, suppose that $F \in \mathcal{C}^\infty(M, \mathbb{R}^l)$ is G -invariant. Using a regular chart (U, φ) , the proof of 3.3.6, together with 3.1.20 shows that on $\pi(U)$ there exists a smooth map \tilde{F}_U such that $\tilde{F}_U \circ \pi = F$ on U . The sets $\pi(U)$ form a covering of M/G by chart neighborhoods (since $\pi(U) = U'$), so we may find a partition of unity (χ_j) on M/G subordinate to it, say with $\text{supp } \chi_j \subseteq U'_j$ for all j . Then setting $\tilde{F} := \sum_j \chi_j \cdot \tilde{F}_{U_j}$ it follows that, for any $x \in M$,

$$\tilde{F}(\pi(x)) = \sum_j \chi_j(\pi(x)) \cdot \tilde{F}_{U_j}(\pi(x)) = \sum_j \chi_j(\pi(x)) \cdot F(x) = F(x).$$

(ii) Suppose first that there exists some $\tilde{F} \in \mathcal{C}^\infty(M/G, \mathbb{R}^l)$ such that $F(x) = 0$ if and only if $\tilde{F}(\pi(x)) = 0$. Then if y is in the same orbit as x , $\pi(x) = \pi(y)$. Thus, if $F(x) = 0$ then $0 = \tilde{F}(\pi(x)) = \tilde{F}(\pi(y))$, which in turn is equivalent to $F(y) = 0$. It follows that S_F is G -invariant.

Conversely, if S_F is G -invariant then by 3.3.8 (together with 3.1.20) it follows that any point in M has a neighborhood U such that there exists some smooth map \tilde{F}_U on $U' = \pi(U)$ with the property that, for all $x \in U$, $F(x) = 0$ if and only if $\tilde{F}_U(\pi(x)) = 0$. Let $\tilde{F}_U(z) = (\tilde{F}_1(z), \dots, \tilde{F}_l(z))$. Then we may replace \tilde{F}_i by $(\tilde{F}_i)^2$ for all $i \in \{1, \dots, l\}$ while retaining the property stated above. Hence without loss of generality we may assume that each \tilde{F}_i is non-negative. As in (i), pick a partition of unity (χ_j) on M/G subordinate to a covering by such neighborhoods U'_j ($j \in \mathbb{N}$) and set $\tilde{F} := \sum_j \chi_j \cdot \tilde{F}_{U_j}$. Let $x \in M$ with $F(x) = 0$ and suppose that $\pi(x) \in U'_j = \pi(U_j)$. Then for some $x_j \in U_j$ we have $\pi(x) = \pi(x_j)$, so since S_F is G -invariant we obtain $x_j \in S_F$ as well. Hence $0 = \tilde{F}_{U_j}(\pi(x_j)) = \tilde{F}_{U_j}(\pi(x))$ for any such j , implying $\tilde{F}(\pi(x)) = 0$. Conversely, suppose that $\tilde{F}(\pi(x)) = 0$ and pick k such that $\chi_k(\pi(x)) > 0$. Then since each \tilde{F}_{U_j} is non-negative, it follows that $\tilde{F}_{U_k}(\pi(x)) = 0$. Since $U'_k = \pi(U_k)$, there exists some $x_k \in U_k$ with $\pi(x_k) = \pi(x)$. By construction of \tilde{F}_{U_k} , therefore, $F(x_k) = 0$, i.e., $x_k \in S_F$. But then also $F(x) = 0$ since S_F is G -invariant.

(iii) Suppose that $N = \pi^{-1}(\tilde{N})$ and let $x \in N$ and $g \in G$ such that $y := g \cdot x$ exists. Then $\pi(y) = \pi(x) \in \tilde{N}$, so $y \in \pi^{-1}(\tilde{N}) = N$. Thus N is G -invariant.

Conversely, if N is a G -invariant regular submanifold of M and $\Phi: \mathcal{U} \rightarrow M$ denotes the action of G on M (cf. 3.1.1), then $\mathcal{U}' := \mathcal{U} \cap (G \times N)$ is open in $G \times N$ and contains $\{e\} \times N$. Also, $\Phi: \mathcal{U}' \rightarrow M$ is smooth. Moreover, $\Phi(\mathcal{U}') \subseteq N$ since N is G -invariant and because N is regular it follows that $\Phi: \mathcal{U}' \rightarrow N$ is smooth, hence is a local transformation group. Let $x \in N$ and let S be the G -orbit of x . Then S is a regular submanifold of M (by 3.1.6) and is contained in N . Thus by [11, 13.6] it is an immersive submanifold of N . Also, it carries the trace topology of M , as does N , so in fact it carries the trace topology of N , i.e., it is a regular submanifold of N . Furthermore, if \mathcal{B} is a neighborhood basis of x in M such that $S' \cap U$ is connected in S' for every orbit S' and every $U \in \mathcal{B}$ then $\{U \cap N \mid U \in \mathcal{B}\}$ has the same property in N (since it carries the trace topology of M), so G acts regularly on N as well. We are therefore in a position to apply 3.1.19 to $\Phi: \mathcal{U}' \rightarrow N$ and obtain that $\pi|_N: N \rightarrow N/G$ is a surjective submersion and $N/G = \pi(N) =: \tilde{N}$ is an $(n - k)$ -dimensional manifold. Since N is G -invariant, if $y \in \pi^{-1}(\tilde{N})$ then $\pi(y) = \pi(x)$ for some $x \in N$, so $y \in N$, i.e., $N = \pi^{-1}(\tilde{N})$.

It only remains to prove that \tilde{N} is a regular submanifold of M/G . To see this, we will proceed analogously to the proof of 2.2.14. Let $x \in N$, let S be the orbit of x (so $S \subseteq N$), and pick $v_1, \dots, v_k \in \mathfrak{g}$ such that $\{v_1(x), \dots, v_k(x)\}$ is a basis of $\Delta_\Phi(x)$. Next, choose an adapted chart $(\chi = (y^1, \dots, y^m), W)$ around x in M such that $((y^1, \dots, y^n), W \cap N)$ is a chart of N , $\chi(x) = 0 \in \mathbb{R}^m$, and $\chi(W \cap N) = \{y^{n+1} =$

$\dots = y^m = 0\}$. Also, we may suppose that $v_1(x), \dots, v_k(x), \frac{\partial}{\partial y^{k+1}} \Big|_x, \dots, \frac{\partial}{\partial y^n} \Big|_x$ is a basis of $T_x N$, and $v_1(x), \dots, v_k(x), \frac{\partial}{\partial y^{k+1}} \Big|_x, \dots, \frac{\partial}{\partial y^m} \Big|_x$ is a basis of $T_x M$. Let

$$f(t^1, \dots, t^m) := (\text{Fl}_{t^1}^{v_1} \circ \dots \circ \text{Fl}_{t^k}^{v_k})(\chi^{-1}(0, \dots, 0, t^{k+1}, \dots, t^m)).$$

Then f is a diffeomorphism from some neighborhood $(-d, d)^m$ of $0 \in \mathbb{R}^m$ onto a neighborhood of x in M , and we take $\varphi := f^{-1}$ on a suitable neighborhood U of x to obtain that (U, φ) is a chart of M around x . Furthermore, since χ is an adapted chart and N is G -invariant,

$$(t^1, \dots, t^n) \mapsto f(t^1, \dots, t^n, 0, \dots, 0)$$

maps into N and can (by making d smaller if necessary) also be assumed to be a diffeomorphism. This means that (φ, U) is also an adapted chart for N .

Since S is an orbit of G ,

$$y \in S \Leftrightarrow \text{Fl}_{t^1}^{v_1} \circ \dots \circ \text{Fl}_{t^k}^{v_k}(y) \in S$$

for all y and t^1, \dots, t^k where the right hand side is defined. Therefore, for any $y = f(t^1, \dots, t^m) \in U$ we have

$$y = f(t^1, \dots, t^m) \in S \Leftrightarrow f(0, \dots, 0, t^{k+1}, \dots, t^m) \in S. \quad (3.3.4)$$

This means that $U \cap S$ is the disjoint union of connected sets of the form

$$U_c := \{y \in U \mid x^{k+1}(y) = c_{k+1}, \dots, x^m(y) = c_m\}$$

where $c = (c_{k+1}, \dots, c_m)$ is constant. Since G acts regularly, we may suppose in addition that $U \cap S$ is connected. Then since $f(0, \dots, 0) = x \in S$ it follows that the only non-empty U_c is U_0 , so $S \cap U = \{y \in U \mid x^{k+1}(y) = \dots = x^m(y) = 0\}$.

Summing up, we have that (φ, U) is a flat chart for G on M and $(\varphi, U \cap N)$ is a flat chart for G on N . Since G acts regularly, the proof of 3.1.17 shows that we may in addition assume that U is so small that (φ, U) is regular for Δ_Φ on M and $(\varphi, U \cap N)$ is regular for Δ_Φ on N . Then by 3.1.20, on $U' = \pi(U)$ we obtain a chart φ' for M/G around $\pi(x)$ by

$$\varphi' : y' \mapsto (x^{k+1}(y), \dots, x^m(y)), \quad U' \rightarrow \mathbb{R}^{m-k}$$

with y any element of $\pi^{-1}(y') \cap U$. By the same reasoning,

$$y' \mapsto (x^{k+1}(y), \dots, x^n(y)), \quad U' \cap \tilde{N} \rightarrow \mathbb{R}^{n-k}$$

(with y any element of $\pi^{-1}(y') \cap U \cap N$) is a chart for $\tilde{N} = N/G$ around $\pi(x)$. This shows that (φ', U') is a chart adapted to \tilde{N} , i.e., \tilde{N} is a regular submanifold of M/G . \square

3.4 Dimensional analysis

In a typical physics problem, there are certain fundamental physical quantities, such as length, time, mass, \dots , which can all be scaled independently of each other. Let z^1, \dots, z^r denote these quantities, then this scaling can be described by the action of a scaling group

$$(z^1, \dots, z^r) \mapsto (\lambda_1 z^1, \dots, \lambda_r z^r),$$

with fixed scaling factors $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{R}^r$. The underlying group therefore is the r -th power of the multiplicative group \mathbb{R}^+ .

Furthermore, there typically are certain derived quantities, such as velocity, acceleration, force, density, \dots , which also scale if the fundamental quantities are scaled. Call these quantities $x = (x^1, \dots, x^m)$. Then the action of the scaling group on the derived quantities takes the form

$$\lambda \cdot (x^1, \dots, x^m) = (\lambda_1^{\alpha_{11}} \lambda_2^{\alpha_{21}} \dots \lambda_r^{\alpha_{r1}} x^1, \dots, \lambda_1^{\alpha_{1m}} \lambda_2^{\alpha_{2m}} \lambda_r^{\alpha_{rm}} x^m), \quad (3.4.1)$$

where α_{ij} , $i = 1, \dots, r$, $j = 1, \dots, m$ is a matrix of real numbers determined by the problem at hand. For example, if z^1 is length, z^2 is time and z^3 is mass, then the action of scaling on velocity x^1 and force x^2 are given by

$$\lambda \cdot (x^1, x^2) = (\lambda_1 \lambda_2^{-1} x^1, \lambda_1^2 \lambda_2^{-2} \lambda_3 x^2).$$

If a derived quantity is invariant under the corresponding scaling then it is called *dimensionless*. To describe some physical situation, one often has functional relations of the form $F(x^1, \dots, x^m) = 0$ for the derived quantities. Such a relation is called *unit-free* if it remains unchanged under a rescaling of the fundamental quantities. It turns out that such unit-free relations are often of great physical significance. The Buckingham Pi-Theorem shows that any unit-free relation can be written solely in terms of dimensionless quantities.

3.4.1 Theorem. (*Buckingham Pi-theorem*) Let z^1, \dots, z^r be fundamental physical quantities that scale independently according to $z^i \mapsto \lambda_i z^i$. Let x^1, \dots, x^m be derived quantities that scale according to (3.4.1) for some $(r \times m)$ -matrix of constants $A = (\alpha_{ij})$ and let s be the rank of A . Then

(i) There exist $m - s$ functionally independent dimensionless ‘power products’

$$\pi^k = (x^1)^{\beta_{1k}} (x^2)^{\beta_{2k}} \dots (x^m)^{\beta_{mk}}, \quad k = 1, \dots, m - s \quad (3.4.2)$$

such that any other dimensionless quantity can be written as a function of π^1, \dots, π^{m-s} . The columns of the matrix $B = (\beta_{jk})$ in (3.4.2) can be taken to be any basis of $\ker A$.

(ii) If $F(x^1, \dots, x^m) = 0$ is any unit-free relation among the given derived quantities, then there is an equivalent relation $\tilde{F} = 0$ which can be expressed solely in terms of the above dimensionless power products:

$$F(x) = 0 \Leftrightarrow \tilde{F}(\pi^1(x), \dots, \pi^{m-s}(x)) = 0.$$

Proof. The proof will basically be an application of 3.3.11. As the underlying manifold M we take the positive octant in \mathbb{R}^m , so

$$M = \{x = (x^1, \dots, x^m) \mid x^i > 0, i = 1, \dots, m\}.$$

Then the multiplicative group $G = (\mathbb{R}^+)^r$ acts globally on M via (3.4.1). Since the action is multiplicative, its generators are found by differentiating (3.4.1) with respect to λ_i and then setting $\lambda_1 = \dots = \lambda_r = 1$:

$$v_i = \alpha_{i1} x^1 \frac{\partial}{\partial x^1} + \alpha_{i2} x^2 \frac{\partial}{\partial x^2} + \dots + \alpha_{im} x^m \frac{\partial}{\partial x^m}, \quad i = 1, \dots, r.$$

It follows that the dimension of the span of v_1, \dots, v_r is precisely the rank of A , namely s , so G has s -dimensional orbits. For a function f to be a global invariant of G , by 3.2.7 it has to satisfy

$$v_i(f) = 0, \quad i = 1, \dots, r.$$

In particular, if $f = \pi_k$ is given by (3.4.2), then this condition is satisfied if and only if the exponents β_{jk} satisfy the system of linear equations

$$\sum_{j=1}^m \alpha_{ij} \beta_{jk} = 0, \quad i = 1, \dots, r. \quad (3.4.3)$$

Since $\text{rk}(A) = s$, there are $m - s$ linearly independent solutions to this system. Choosing these solutions for the β_{jk} , let $\pi := (\pi^1, \dots, \pi^{m-s})$. Then the Jacobian of π at $x = (1, \dots, 1)$ is the transpose of the matrix (β_{jk}) , $j = 1, \dots, m$, $k = 1, \dots, m - s$, and its rank therefore is $m - s$. For any $a > 0$ we have (as matrix-multiplication)

$$\pi(ax^1, x^2, \dots, x^m) = \text{diag}(a^{\beta_{11}}, \dots, a^{\beta_{1(m-s)}}) \cdot \pi(x^1, x^2, \dots, x^m),$$

so since $D\pi(1, 1, \dots, 1)$ has a non-zero $(m - s) \times (m - s)$ -sub-determinant, the same is true of $D\pi(a, 1, \dots, 1)$. Analogously we can argue for varying the x^2 -component, and so on. Altogether, it follows that the rank of the Jacobian of π is $(m - s)$ everywhere on M . Thus by (the easy direction of) 3.3.5, $(\pi^1, \dots, \pi^{m-s})$ is a functionally independent system of invariants of G on M .

Next we show that the orbits of G are precisely the level sets of the function π . To see this, let x, \tilde{x} be any two points in M . Then by the definition of M there exist exponents $t_j, j = 1, \dots, m$ such that $x^j = e^{t_j} \cdot \tilde{x}^j$ for all j . Therefore

$$\begin{aligned} \pi^k(x) &= e^{\beta_{1k}t_1} e^{\beta_{2k}t_2} \dots e^{\beta_{mk}t_m} (\tilde{x}^1)^{\beta_{1k}} \dots (\tilde{x}^m)^{\beta_{mk}} = \pi^k(\tilde{x}) \\ &\Leftrightarrow \sum_{j=1}^m \beta_{jk} t_j = 0, \end{aligned}$$

so x and \tilde{x} lie in the same level set of π if and only if the t_j satisfy

$$\sum_{j=1}^m \beta_{jk} t_j = 0, \quad k = 1, \dots, m - s. \quad (3.4.4)$$

By construction, the columns of the $m \times (m - s)$ -matrix $B := (\beta_{jk})$ form a basis of $\ker A$, so in particular $A \cdot B = 0$. Moreover, (3.4.4) means that $B^\top \cdot t = 0$, where $t = (t^1, \dots, t^m)^\top$, i.e., $t \in \ker(B^\top)$. We claim that this, in turn, is equivalent to the existence of s_1, \dots, s_r with

$$t_j = \sum_{i=1}^r s_i \alpha_{ij}, \quad j = 1, \dots, m \quad (3.4.5)$$

i.e., to $t \in \text{im}(A^\top)$. This means we have to show that $\text{im}(A^\top) = \ker(B^\top)$. Now if $w = A^\top u \in \text{im}(A^\top)$, then $B^\top w = B^\top A^\top u = (AB)^\top u = 0$, so $\text{im}(A^\top) \subseteq \ker(B^\top)$. On the other hand, $\dim \text{im}(A^\top) = \dim \text{im}(A) = s$, and also $\dim \ker(B^\top) = m - \dim \text{im}(B) = m - \dim \text{im}(B) = s$, so indeed we have equality.

Hence two points x and \tilde{x} in M have the same image under π if and only if $x = \lambda \tilde{x}$, where $\lambda_i = e^{s_i}$, i.e., if and only if they lie in the same orbit under G . Since π has maximal rank $m - s$ everywhere, its level sets are regular submanifolds of dimension s , so G has s -dimensional regular submanifolds as orbits. Moreover, the proof of 3.3.8 shows that, around any point, π^1, \dots, π^{m-s} can be completed by a set of s parametric variables to form a flat chart for the group action of G . In the present situation, such parametric variables can even be chosen globally on all of M : indeed, pick numbers $\beta_{jk}, j = 1, \dots, m, k = m - s + 1, \dots, m$ that supplement the matrix B from above to obtain an invertible $m \times m$ matrix \tilde{B} and set

$$\tilde{\pi}(x) := \left(\prod_{i=1}^m (x^i)^{\beta_{i1}}, \dots, \prod_{i=1}^m (x^i)^{\beta_{im}} \right).$$

Then the first k components of $\tilde{\pi}$ are precisely π , and the remaining $m - s$ components are our new parametric variables. By the same reasoning as above for π it follows that $\tilde{\pi}$ has rank m everywhere on M , hence is a local diffeomorphism. In addition, $\tilde{\pi}$ is injective: to see this, note first that $\tilde{\pi}(x) = \tilde{\pi}(y)$ implies $x = y$ if and only if $\tilde{\pi}^i(x) = 1$ implies $x^i = 1$ for all i . Now if $\tilde{\pi}^i(x) = 1$ for all i then taking logarithms it follows that $\tilde{B}^\top \cdot (\ln(x^1), \dots, \ln(x^m))^\top = 0$, which due to the invertibility of \tilde{B} indeed implies $x^i = 1$ for all i . Thus we obtain the desired global chart $\tilde{\pi}$ for M .

Any orbit intersects this chart in a single slice ($\pi = \text{const.}$), so the action of G is regular. Moreover, by 3.1.20, π induces global coordinates on M/G , hence can be identified with the quotient map $M \rightarrow M/G$ (and M/G can be identified with \mathbb{R}_+^{m-s}). We are therefore in the position to apply 3.3.11. Since being G -invariant by definition is the same as being dimensionless, (i) follows from 3.3.11 (i). Moreover, an equation is unit-free if and only if it is G -invariant, so (ii) follows from 3.3.11 (ii). \square

3.4.2 Example. The energy yield of the first atomic explosion. In 1947, when the amount of energy released by the first atomic explosion was still classified, G. Taylor calculated this energy using dimensional analysis (see [19], our presentation follows [2]).

We model the explosion by the radius R of the spherical fireball emanating from the point of explosion. We assume that R is a function

$$R = F(x^1, x^2, x^3, x^4), \quad (3.4.6)$$

where

$x^1 = E$, the energy released by the explosion,

$x^2 = t$, the time elapsed since the explosion,

$x^3 = \rho_0$, the initial ambient air density,

$x^4 = P_0$, the initial ambient air pressure.

R itself also is a derived quantity, so we set $x^5 = R$. It follows that for this problem $m = 5$. The fundamental physical quantities needed to describe the derived quantities are $z^1 = \text{length}$, $z^2 = \text{mass}$, and $z^3 = \text{time}$, so $r = 3$. The matrix $A = (\alpha_{ij})_{i=1, \dots, 5}^{j=1, \dots, 3}$ has the form

$$A = \begin{pmatrix} 2 & 0 & -3 & -1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ -2 & 1 & 0 & -2 & 0 \end{pmatrix}$$

E.g., energy has dimension $\text{length}^2 \times \text{mass} / \text{time}^2$, hence the first column. The rank of this matrix is $s = 3$, so $m - s = 5 - 3 = 2$. Therefore, by 3.4.1 there are two functionally independent dimensionless power products π^1, π^2 , in terms of which any dimensionless quantity can be described. By (3.4.2), these power products are of the form

$$\begin{aligned} \pi^1(x) &= (x^1)^{\beta_{11}} (x^2)^{\beta_{21}} (x^3)^{\beta_{31}} (x^4)^{\beta_{41}} (x^5)^{\beta_{51}} \\ \pi^2(x) &= (x^1)^{\beta_{12}} (x^2)^{\beta_{22}} (x^3)^{\beta_{32}} (x^4)^{\beta_{42}} (x^5)^{\beta_{52}}. \end{aligned}$$

From the proof of 3.4.1 we know that the 5×2 matrix $B = (\beta_{jk})$ can be constructed by finding a basis of $\ker A$. It follows that

$$B = \begin{pmatrix} -2 & -1 \\ 6 & -2 \\ -3 & 1 \\ 5 & 0 \\ 0 & 5 \end{pmatrix},$$

so

$$\begin{aligned} \pi^1(x) &= (x^1)^{-2}(x^2)^6(x^3)^{-3}(x^4)^5 = \frac{P_0^5 t^6}{E^2 \rho_0^3} \\ \pi^2(x) &= (x^1)^{-1}(x^2)^{-2}(x^3)(x^5)^5 = \frac{R^5 \rho_0}{Et^2}. \end{aligned}$$

By 3.4.1 (ii), the relation (3.4.6) can equivalently be expressed in the form $\tilde{F}(\pi^1, \pi^2) = 0$, and we assume that in fact we can solve for π^2 . Thus there exists a smooth function \tilde{g} of π^1 such that (3.4.6) is equivalent to $\pi^2 = \tilde{g}(\pi^1)$, i.e., to

$$R = \left(\frac{Et^2}{\rho_0} \right)^{1/5} g(\pi^1). \quad (3.4.7)$$

for some function g . By continuity, $g(\pi^1) \approx g(0)$ for π^1 small, so Taylor derived the approximative formula

$$R = At^{2/5}, \quad \text{with } A = \left(\frac{E}{\rho_0} \right)^{1/5} g(0).$$

Here, ρ_0 is known, and to find $g(0)$ one can plot $\log_{10} R$ versus $\log_{10} t$ using data from experiments with conventional explosives (whose E is known). Taylor then used motion picture records of the first atomic explosion, due to J.E. Mack, to plot $5/2 \log_{10} R$ versus $\log_{10} t$ (in c.g.s.-units, but as we know any other system of units would have given the same result!), to obtain an accurate estimate for E :

$$E \approx 7, 14 \times 10^{20} \text{ ergs} = 7, 14 \times 10^{13} J,$$

which corresponds to the energy release of about 16800 tons of TNT.

3.5 Groups and differential equations

We now want to start applying symmetry methods to differential equations. To this end we consider a system \mathcal{S} of differential equations involving p independent variables $x = (x^1, \dots, x^p)$ and q dependent variables $u = (u^1, \dots, u^q)$. Solutions of \mathcal{S} will be functions $u = f(x)$, or, in components,

$$u^\alpha = f^\alpha(x^1, \dots, x^p) \quad (\alpha = 1, \dots, q).$$

Henceforth we will use the convention of using Latin indices for the independent variables and Greek indices for the dependent ones. We let $X = \mathbb{R}^p$ be the space of independent variables and $U = \mathbb{R}^q$ the space of dependent variables. Then basically a symmetry group of \mathcal{S} will be a local transformation group G acting on $M := X \times U$ in such a way that it 'transforms solutions of \mathcal{S} into solutions of \mathcal{S} '.

We first need to clarify how such a group of transformations is to act on a function f . The key idea here is to identify $f: X \supseteq \Omega \rightarrow U$ with its graph

$$\Gamma_f = \{(x, f(x)) \mid x \in \Omega\}.$$

Then Γ_f is a p -dimensional submanifold of M . If $g \in G$ and $\Gamma_f \subseteq \mathcal{U}_g$, the domain of g , then g acts on Γ_f by

$$g \cdot \Gamma_f := \{(\tilde{x}, \tilde{u}) = g \cdot (x, u) \mid (x, u) \in \Gamma_f\}.$$

Note that $g \cdot \Gamma_f$ will in general no longer be the graph of a function. However, since G acts smoothly and $e \cdot \Gamma_f = \Gamma_f$, by shrinking Ω and restricting to elements of G near to e we can always achieve that $g \cdot \Gamma_f = \Gamma_{\tilde{f}}$, where \tilde{f} (corresponding to $\tilde{u} = \tilde{f}(\tilde{x})$) is a well-defined function. In this case we write $\tilde{f} = g \cdot f$ and call \tilde{f} the transform of f by g .

3.5.1 Example. Let $X = \mathbb{R}$, $U = \mathbb{R}$, so $p = q = 1$ and again consider the rotation group $G = SO(2)$ on $M = X \times U = \mathbb{R}^2$. The action of G is given by

$$(\tilde{x}, \tilde{u}) = \varepsilon \cdot (x, u) = (x \cos \varepsilon - u \sin \varepsilon, x \sin \varepsilon + u \cos \varepsilon) \quad (3.5.1)$$

Now if $f: x \mapsto u = f(x)$ is a function then G acts on f by rotating its graph, and in general this rotated graph will not be the graph of a well-defined function: think of a straight line or a parabola. The example of the parabola in particular demonstrates that it will in general not suffice to choose g near e (i.e., ε near 0) but that one has to also restrict the domain of f . But for $|\varepsilon|$ small and restricting f to a sufficiently small domain we do obtain a well defined function $\varepsilon \cdot f = \tilde{f}$.

To explicitly calculate the action of G on a linear function, let $u = f(x) = ax + b$. Then any point in Γ_f is of the form $(x, ax + b)$, which is rotated by ε to the point

$$(\tilde{x}, \tilde{u}) = (x \cos \varepsilon - (ax + b) \sin \varepsilon, x \sin \varepsilon + (ax + b) \cos \varepsilon).$$

To determine $\tilde{u} = \tilde{f}(\tilde{x})$ we have to eliminate x from this equation, which is possible for ε small (so that $\cot \varepsilon \neq a$). Then

$$x = \frac{\tilde{x} + b \sin \varepsilon}{\cos \varepsilon - a \sin \varepsilon},$$

and therefore

$$\tilde{u} = \tilde{f}(\tilde{x}) = \frac{\sin \varepsilon + a \cos \varepsilon}{\cos \varepsilon - a \sin \varepsilon} \tilde{x} + \frac{b}{\cos \varepsilon - a \sin \varepsilon},$$

which, as expected, is again a linear function.

The general procedure for calculating $\tilde{f} = g \cdot f$ from f is as follows. Write

$$(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x, u), \Phi_g(x, u)), \quad (3.5.2)$$

with Ξ_g, Φ_g smooth. The graph $\Gamma_{\tilde{f}} = g \cdot \Gamma_f$ of $\tilde{f} = g \cdot f$ is then given parametrically by the equations (for $x \in \Omega$)

$$\begin{aligned} \tilde{x} &= \Xi_g(x, f(x)) = \Xi_g \circ (\text{id} \times f)(x) \\ \tilde{u} &= \Phi_g(x, f(x)) = \Phi_g \circ (\text{id} \times f)(x) \end{aligned}$$

To calculate \tilde{f} we have to eliminate x from these equations. For $g = e$, $\Xi_e \circ (\text{id} \times f) = \text{id}$, hence for g in a neighborhood of e the Jacobian of $\Xi_g \circ (\text{id} \times f)$ is nonsingular and so by the inverse function theorem we can locally solve for x :

$$x = [\Xi_g \circ (\text{id} \times f)]^{-1}(\tilde{x}).$$

Consequently,

$$g \cdot f = [\Phi_g \circ (\text{id} \times f)] \circ [\Xi_g \circ (\text{id} \times f)]^{-1}. \quad (3.5.3)$$

3.5.2 Lemma. *Let f be a local smooth map, $f: \Omega \rightarrow U$, and let $g, h \in G$. Then $h \cdot (g \cdot f) = (h \cdot g) \cdot f$, wherever defined.*

Proof. Since any function is uniquely determined by its graph it suffices to note that

$$\Gamma_{(h \cdot g) \cdot f} = (h \cdot g) \Gamma_f = h \cdot (g \cdot \Gamma_f) = h \cdot \Gamma_{g \cdot f} = \Gamma_{h \cdot (g \cdot f)}.$$

□

An important special case where \tilde{f} is automatically defined unrestrictedly is that of a *projectable* group action. For such actions, Ξ_g is a function of x only, i.e.,

$$(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x), \Phi_g(x, u)),$$

and so $\tilde{f}(\tilde{x}) = [\Phi_g \circ (\text{id} \times f)] \circ \Xi_g^{-1}(\tilde{x})$. Using this terminology, we can now define:

3.5.3 Definition. *Let \mathcal{S} be a system of differential equations. A symmetry group of \mathcal{S} is a local transformation group G acting on an open subset of $M = X \times U$ such that whenever f is a solution of \mathcal{S} and $g \cdot f$ is defined for some $g \in G$ then also $g \cdot f$ is a solution of \mathcal{S} . (Here solution means any smooth solution defined on any subdomain Ω of X .)*

3.5.4 Example. (i) Let \mathcal{S} consist of the equation $u_{xx} = 0$. The solutions are precisely the linear functions on \mathbb{R} . Since $G = SO(2)$ transforms linear functions to linear functions, it is a symmetry group of \mathcal{S} .

(ii) Let $\mathcal{S} = \{u_t = u_{xx}\}$, the heat equation, and consider the group action

$$(x, t, u) \mapsto (x + \varepsilon a, t + \varepsilon b, u) \quad (\varepsilon \in \mathbb{R}).$$

This is a symmetry group of \mathcal{S} since $f(x - \varepsilon a, t - \varepsilon b)$ is a solution of the heat equation whenever f is.

One immediate advantage of knowing the symmetry group of a differential equation is that it gives a straightforward way of constructing new solutions from known ones (sometimes even from trivial ones, e.g., constant solutions) simply by applying the group transformations. Our aim ultimately is to derive infinitesimal criteria for a local transformation group to be a symmetry group of a differential equation, which can then be used to explicitly calculate the full symmetry group of any given differential equation. For this purpose we first need to develop some more technical machinery.

3.6 Prolongation

In the previous section we identified any smooth function with its graph in order to define the action of a transformation group on it. Since we are now interested in differential equations we need to find a way of simultaneously considering a function and its derivatives up to a certain order. The mechanism we are going to employ is that of prolongation, a simplified version of the theory of jet bundles.

Given a smooth function $f: \mathbb{R}^p \rightarrow \mathbb{R}$, $x \mapsto f(x^1, \dots, x^p)$ and any $k \in \mathbb{N}_0$, we let $J = (j_1, \dots, j_k)$ be an unordered k -tuple with $1 \leq j_l \leq p$ for all l . The order of this tuple is defined as $\#J = k$, and the corresponding partial derivative of f is

$$\partial_J f(x) = \frac{\partial^k f}{\partial x^{j_1} \partial x^{j_2} \dots \partial x^{j_k}}.$$

There are

$$p_k := \binom{p+k-1}{k}$$

different possible k -th order partial derivatives of f . To see this, note that since the order of derivatives is irrelevant we may first bring J in ascending order, and then uniquely describe any possible choice by writing $*$ for each number that appears in J and $|$ to signify that we increase the number. For example, if $p = 5$ and $k = 2$, the string $**|||$ corresponds to the tuple $(1, 1)$, $|*|*||$ to $(2, 3)$, and $(|||*|*)$ to $(4, 5)$, etc. We need k stars and $p - 1$ vertical lines, so our problem is equivalent to determining the number of possible selections of k elements from a $p - 1 + k$ -element set, hence the above formula.

If $f: X \rightarrow U$ with $X = \mathbb{R}^p$ and $U = \mathbb{R}^q$, so $u = f(x) = (f^1(x), \dots, f^q(x))$, then we need $q \cdot p_k$ numbers $u_J^\alpha = \partial_J f^\alpha(x)$ to represent all possible k -th order derivatives of f at any point x . Therefore, we set $U_k := \mathbb{R}^{q \cdot p_k}$ and write u_J^α , with $\alpha \in \{1, \dots, q\}$ and J any unordered multi-index as above. Moreover, we let

$$U^{(n)} := U \times U_1 \times \dots \times U_n.$$

The coordinates of $U^{(n)}$ can be used to represent all the derivatives of any map $f: X \rightarrow U$ of orders 0 to n . The dimension of $U^{(n)}$ is

$$q + qp_1 + \dots + qp_n = q \binom{p+n}{n} =: qp^{(n)}.$$

We will denote elements of $U^{(n)}$ by $u^{(n)}$, with $q \cdot p^{(n)}$ components u_J^α , with $\alpha \in \{1, \dots, q\}$ and $J = (j_1, \dots, j_k)$ an unordered multiindex, $1 \leq j_l \leq p$, $0 \leq k \leq n$. For $k = 0$ there is only one such multiindex, denoted by 0 and u_0^α is the component u^α of u .

3.6.1 Example. Let $p = 2$, $q = 1$. Then $X = \mathbb{R}^2$ has coordinates $(x^1, x^2) = (x, y)$, and $U = \mathbb{R}$ has the coordinate u . U_1 equals \mathbb{R}^2 , with coordinates (u_x, u_y) , representing all first order derivatives of u . Also, $U_2 = \mathbb{R}^3$ has coordinates (u_{xx}, u_{xy}, u_{yy}) , representing all second order derivatives of u . In general, $U_k = \mathbb{R}^{k+1}$ since there are $k + 1$ derivatives of u of order k , namely $\frac{\partial^k u}{\partial x^i \partial y^{k-i}}$, $i = 0, \dots, k$. Also, $U^{(2)} = U \times U_1 \times U_2 = \mathbb{R}^6$, with coordinates $u^{(2)} = (u; u_x, u_y; u_{xx}, u_{xy}, u_{yy})$, representing all derivatives of u with respect to x and y of order at most 2.

Now for any smooth function $f: X \rightarrow U$ we define its n -th prolongation $\text{pr}^{(n)} f: X \rightarrow U^{(n)}$, $u^{(n)} = \text{pr}^{(n)} f(x)$ by

$$u_J^\alpha = \partial_J f^\alpha(x).$$

Thus for any x , $\text{pr}^{(n)} f(x)$ is a vector with $q \cdot p^{(n)}$ entries representing the values of f and all its derivatives up to order n at x . In this sense we might also identify $\text{pr}^{(n)} f(x)$ with the n -th Taylor polynomial of f at x .

3.6.2 Example. For $p = 2$, $q = 1$, we have $u = f(x, y)$. Then the second prolongation of f , $u^{(2)} = \text{pr}^{(2)} f(x, y)$ is given by

$$(u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}) = (f; \partial_x f, \partial_y f, \partial_x^2 f, \partial_{xy} f, \partial_y^2 f),$$

evaluated at (x, y) .

The space $X \times U^{(n)}$ is also called the n -th order jet-space of $X \times U$, and $\text{pr}^{(n)} f(x)$ is called the n -jet of f at x . If M is an open subset of $X \times U$, then we set

$$M^{(n)} := M \times U_1 \times \dots \times U_n.$$

If $u = f(x)$ is a function whose graph lies in M then the n -th prolongation $\text{pr}^{(n)} f$ is a function whose graph lies in $M^{(n)}$.

3.7 Systems of differential equations

For a system \mathcal{S} of n -th order differential equations in p independent and q dependent variables we write

$$P_\nu(x, u^{(n)}) = 0 \quad \nu = 1, \dots, l,$$

where $x = (x^1, \dots, x^p)$, $u = (u^1, \dots, u^q)$. Here the functions

$$P(x, u^{(n)}) = (P_1(x, u^{(n)}), \dots, P_l(x, u^{(n)}))$$

are assumed to be smooth, i.e., $P \in \mathcal{C}^\infty(X \times U^{(n)}, \mathbb{R}^l)$. The differential equations in \mathcal{S} determine where this map vanishes on $X \times U^{(n)}$, i.e., they determine a corresponding subvariety

$$\mathcal{S}_P = \{(x, u^{(n)}) \mid P(x, u^{(n)}) = 0\} \subseteq X \times U^{(n)}$$

of the total jet space $X \times U^{(n)}$. We may therefore identify the system \mathcal{S} with this set \mathcal{S}_P , and we shall do so at many places below.

A solution of \mathcal{S} is a smooth function $u = f(x)$ such that

$$P_\nu(x, \text{pr}^{(n)}f(x)) = 0, \quad \nu = 1, \dots, l,$$

for all x in the domain of f . Geometrically, this means that the graph of the prolongation $\text{pr}^{(n)}f$ is contained in \mathcal{S}_P :

$$\Gamma_f^{(n)} \equiv \{(x, \text{pr}^{(n)}f(x))\} \subseteq \mathcal{S}_P = \{P(x, u^{(n)}) = 0\}.$$

3.7.1 Example. Consider the Laplace equation in two variables:

$$u_{xx} + u_{yy} = 0. \tag{3.7.1}$$

Here, $p = 2$, $q = 1$, and $n = 2$ since the equation is of second order. The coordinates of $X \times U^{(2)}$ are $(x, y; u; u_x, u_y; u_{xx}, u_{xy}, u_{yy})$, and (3.7.1) describes a hyperplane \mathcal{S}_P in $X \times U^{(2)}$. A function f is a solution of (3.7.1) if the graph of $\text{pr}^{(2)}f$ is contained in \mathcal{S}_P . For example, let $f(x, y) = x^3 - 3xy^2$. Then

$$(x, y, \text{pr}^{(2)}f(x, y)) = (x, y; x^3 - 3xy^2; 3x^2 - 3y^2, -6xy, -6x) \in \mathcal{S}_P.$$

3.8 Prolongation of group actions

In this section, given a local transformation group G on an open subset M of $X \times U$, we will construct an induced action $\text{pr}^{(n)}G$, the n -th prolongation of G , on $M^{(n)}$. The idea is to define $\text{pr}^{(n)}G$ in such a way that it transforms the derivatives of any smooth function $u = f(x)$ into the corresponding derivatives of the transformed function $\tilde{u} = \tilde{f}(\tilde{x})$.

Thus let $(x_0, u_0^{(n)}) \in M^{(n)}$ and choose any smooth function $u = f(x)$ defined in a neighborhood of x_0 whose graph lies in M and such that it has the given derivatives at x_0 , i.e., such that $u_0^{(n)} = \text{pr}^{(n)}f(x_0)$. To see that such an f indeed exists, one may take it to be an appropriate Taylor polynomial:

$$f^\alpha(x) := \sum_J \frac{u_{J0}^\alpha}{J!} (x - x_0)^J, \quad \alpha = 1, \dots, q. \tag{3.8.1}$$

Here the sum is over all $J = (j_1, \dots, j_k)$ with $0 \leq k \leq n$, and

$$(x - x_0)^J := (x^{j_1} - x_0^{j_1})(x^{j_2} - x_0^{j_2}) \dots (x^{j_k} - x_0^{j_k}).$$

Also, for given J we set $\tilde{J} := (\tilde{j}_1, \dots, \tilde{j}_p)$, where \tilde{j}_i equals the number of j_κ 's which equal i . For example, if $J = (1, 1, 1, 2, 4, 4)$, $p = 4$, $k = 6$, then $\tilde{J} = (3, 1, 0, 2)$. Finally, $\tilde{J} = \tilde{j}_1! \dots \tilde{j}_p!$.

If g is close to e then the transformed function $\tilde{f} = g \cdot f$ (as in (3.5.3)) is defined in a neighborhood of $(\tilde{x}_0, \tilde{u}_0) = g \cdot (x_0, u_0)$, where $u_0 = f(x_0)$. We then define

$$\text{pr}^{(n)}g \cdot (x_0, u_0^{(n)}) = (\tilde{x}_0, \tilde{u}_0^{(n)}),$$

where

$$\tilde{u}_0^{(n)} = \text{pr}^{(n)}(g \cdot f)(\tilde{x}_0). \quad (3.8.2)$$

An important point to note here is that, due to the chain rule, the value of $\text{pr}^{(n)}g \cdot (x_0, u_0^{(n)})$ depends exclusively on the derivatives of f at the point x_0 up to order n , i.e., only on $(x_0, u_0^{(n)})$ itself. It is therefore independent of the choice of the auxiliary function f from above and therefore provides a well-defined operation on $M^{(n)}$.

3.8.1 Lemma. *For any local transformation group G on M as above, $\text{pr}^{(n)}G$ is a local transformation group on $M^{(n)}$.*

Proof. We first note that $(g, (x_0, u_0^{(n)})) \mapsto \text{pr}^{(n)}g \cdot (x_0, u_0^{(n)})$ is smooth. Indeed, taking the f from (3.8.1) this follows directly from (3.5.3). To see this, it suffices to note that the inverse function theorem yields an inverse that automatically depends smoothly on all parameters that the original function depends upon (in our case, the $u_0^{(n)}$): if $f = f(x, \alpha)$ (α representing any parameters), then f^{-1} is the solution of the implicit equation $F(x, y, \alpha) = f(y, \alpha) - x = 0$, so the implicit function theorem yields the claim.

It remains to verify (i) and (ii) from 3.1.1. Here, (ii) is automatic because for $g = e$ we get $g \cdot f = f$. Thus we are left with proving that

$$\text{pr}^{(n)}h(\text{pr}^{(n)}g \cdot (x_0, u_0^{(n)})) = \text{pr}^{(n)}(h \cdot g)(x_0, u_0^{(n)}). \quad (3.8.3)$$

To see this, let $(x_0, u_0^{(n)}) \in M^{(n)}$ and pick a smooth function f defined near x_0 with $u_0^{(n)} = \text{pr}^{(n)}f(x_0)$. Then

$$(\tilde{x}_0, \tilde{u}_0^{(n)}) = g \cdot (x_0, u_0^{(n)}) = (\tilde{x}_0, \text{pr}^{(n)}(g \cdot f)(\tilde{x}_0)).$$

To determine the left hand side of (3.8.3) we have to find any smooth function F such that $\text{pr}^{(n)}F(\tilde{x}_0) = \text{pr}^{(n)}(g \cdot f)(\tilde{x}_0)$, calculate $h \cdot F$ and then determine $\text{pr}^{(n)}h \cdot F$ at the corresponding point. Naturally, we take $F := g \cdot f$. Then using the terminology from (3.5.2) we obtain

$$\text{pr}^{(n)}h(\text{pr}^{(n)}g \cdot (x_0, u_0^{(n)})) = (\Xi_h(\tilde{x}_0, (g \cdot f)(\tilde{x}_0)), \text{pr}^{(n)}(h \cdot F)(\Xi_h(\tilde{x}_0, (g \cdot f)(\tilde{x}_0))))$$

Here,

$$\begin{aligned} \Xi_h(\tilde{x}_0, (g \cdot f)(\tilde{x}_0)) &= \Xi_h(g \cdot (x_0, u_0)) = \text{proj}_1(h \cdot (g \cdot (x_0, u_0))) \\ &= \text{proj}_1((h \cdot g) \cdot (x_0, u_0)) = \Xi_{h \cdot g}(x_0, u_0), \end{aligned}$$

and, by 3.5.2, $h \cdot F = h \cdot (g \cdot f) = (h \cdot g) \cdot f$, so altogether we get

$$\begin{aligned} \text{pr}^{(n)}h(\text{pr}^{(n)}g \cdot (x_0, u_0^{(n)})) &= (\Xi_{h \cdot g}(x_0, u_0), \text{pr}^{(n)}((h \cdot g)f)(\Xi_{h \cdot g}(x_0, u_0))) \\ &= \text{pr}^{(n)}(h \cdot g)(x_0, u_0^{(n)}). \end{aligned}$$

□

3.8.2 Example. Again let $G = SO(2)$ be the rotation group on $M = \mathbb{R}^2 = X \times U$. We determine the first prolongation of G . We have $p = q = 1$, so $X \times U^{(1)} = \mathbb{R}^3$, with coordinates (x, u, u_x) . If $u = f(x)$ is a local smooth function then $\text{pr}^{(1)}f(x) = (f(x), f'(x))$. Now let $(x^0, u^0, u_x^0) \in X \times U^{(1)}$ and let ε be an angle of rotation. We wish to determine

$$\text{pr}^{(1)}\varepsilon \cdot (x^0, u^0, u_x^0) = (\tilde{x}^0, \tilde{u}^0, \tilde{u}_x^0).$$

To this end, let f be the Taylor polynomial

$$f(x) = u^0 + u_x^0(x - x^0) = u_x^0 \cdot x + (u^0 - u_x^0 x^0),$$

which satisfies $f(x^0) = u^0$, $f'(x^0) = u_x^0$. Then by 3.5.1 we get

$$\tilde{f}(\tilde{x}) = (\varepsilon \cdot f)(\tilde{x}) = \frac{\sin \varepsilon + u_x^0 \cos \varepsilon}{\cos \varepsilon - u_x^0 \sin \varepsilon} \tilde{x} + \frac{u^0 - u_x^0 x^0}{\cos \varepsilon - u_x^0 \sin \varepsilon},$$

which is well-defined for $u_x^0 \neq \cot \varepsilon$. By (3.5.1) we have $\tilde{x}^0 = x^0 \cos \varepsilon - u^0 \sin \varepsilon$, so either again by (3.5.1) or by inserting we get

$$\tilde{u}^0 = \tilde{f}(\tilde{x}^0) = x^0 \sin \varepsilon + u^0 \cos \varepsilon.$$

Finally, we get for the first derivative

$$\tilde{u}_x^0 = \tilde{f}'(\tilde{x}^0) = \frac{\sin \varepsilon + u_x^0 \cos \varepsilon}{\cos \varepsilon - u_x^0 \sin \varepsilon}.$$

Combining this and dropping the 0-superscripts we obtain the action of $\text{pr}^{(1)}SO(2)$ on $X \times U^{(1)}$:

$$\text{pr}^{(1)}\varepsilon \cdot (x, u, u_x) = \left(x \cos \varepsilon - u \sin \varepsilon, x \sin \varepsilon + u \cos \varepsilon, \frac{\sin \varepsilon + u_x \cos \varepsilon}{\cos \varepsilon - u_x \sin \varepsilon} \right),$$

defined for $|\varepsilon| < |\text{arccot}(u_x)|$. The important point to note is that although the action of $SO(2)$ is linear and globally defined, its first prolongation is nonlinear and only locally defined.

In the previous example we see that the first prolongation of G acts on the original variables (x, u) in exactly the same way as G itself. This is actually a general phenomenon. In fact, it follows directly from the definition of $\text{pr}^{(n)}G$ via evaluating derivatives of \tilde{f} that the action of $\text{pr}^{(n)}g$ on $(x, u^{(k)})$ with $k \leq n$ coincides with that of $\text{pr}^{(k)}$. In particular, $\text{pr}^{(0)}G = G$. To give a concise formulation of this property we introduce the natural projections

$$\begin{aligned} \pi_k^n : M^{(n)} &\rightarrow M^{(k)} \\ \pi_k^n(x, u^{(n)}) &= (x, u^{(k)}), \end{aligned}$$

where (for $k \leq n$), $u^{(k)}$ consists of the components u_J^α , $\#J \leq k$ of $u^{(n)}$. For example, if $p = 2$, $q = 1$, then

$$\begin{aligned} \pi_0^2(x, y; u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}) &= (x, y; u) \\ \pi_1^2(x, y; u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}) &= (x, y; u; u_x, u_y). \end{aligned}$$

Then for any $k \leq n$ and any $g \in G$ we have

$$\pi_k^n \circ \text{pr}^{(n)}g = \text{pr}^{(k)}g. \quad (3.8.4)$$

In particular, this observation allows to calculate prolongations of group actions inductively.

3.9 Invariance of differential equations

In Section 3.7 we have identified any given system \mathcal{S} of differential equations $P_\nu(x, u^{(n)}) = 0$ ($\nu = 1, \dots, l$) with a subvariety \mathcal{S}_P of $M^{(n)}$. This identification will allow us to utilize the methods of symmetry group analysis of algebraic equations in the context of differential equations. The essential translation result between the two realms is the following theorem

3.9.1 Theorem. *Let M be an open subset of $X \times U$ and let \mathcal{S} :*

$$P_\nu(x, u^{(n)}) = 0 \quad \nu = 1, \dots, l$$

be an n -th order system of differential equations defined on M , with corresponding subvariety \mathcal{S}_P of $M^{(n)}$. Suppose that G is a local transformation group acting on M whose prolongation leaves \mathcal{S}_P invariant, i.e.,

$$(x, u^{(n)}) \in \mathcal{S}_P \Rightarrow \text{pr}^{(n)}g \cdot (x, u^{(n)}) \in \mathcal{S}_P,$$

whenever defined. Then G is a symmetry group of the system \mathcal{S} in the sense of 3.5.3.

Proof. Let $u = f(x)$, $f: \Omega \rightarrow U$ be a local solution of $P(x, u^{(n)}) = 0$. This means that the graph

$$\Gamma_f^{(n)} = \{(x, \text{pr}^{(n)}f(x)) \mid x \in \Omega\}$$

of $\text{pr}^{(n)}f$ is contained in \mathcal{S}_P . If $g \in G$ is such that $g \cdot f$ is well-defined then, by the very definition of $\text{pr}^{(n)}g$, the graph of $\text{pr}^{(n)}(g \cdot f)$ is given by the image of the graph of $\text{pr}^{(n)}f$ under $\text{pr}^{(n)}g$, i.e.,

$$\Gamma_{g \cdot f}^{(n)} = (\text{pr}^{(n)}g) \cdot \Gamma_f^{(n)}.$$

Since \mathcal{S}_P is invariant under $\text{pr}^{(n)}g$, it follows from this that the graph of $\text{pr}^{(n)}(g \cdot f)$ is contained in \mathcal{S}_P as well. But this just means that $g \cdot f$ is a local solution of the system \mathcal{S} . \square

3.10 Prolongation of vector fields

To make the calculation of symmetry groups of differential equations accessible to the infinitesimal methods we developed for algebraic equations we need to be able to effectively calculate the infinitesimal generators of prolonged group actions.

3.10.1 Definition. *Let $M \subseteq X \times U$ be open and suppose that v is the infinitesimal generator of a local one-parameter transformation group $\varepsilon \mapsto \text{Fl}_\varepsilon^v$. Then the n -th prolongation of v , denoted by $\text{pr}^{(n)}v$, is the local vector field on $M^{(n)}$ which is the infinitesimal generator of the corresponding prolonged one-parameter group $\text{pr}^{(n)}\text{Fl}_\varepsilon^v$:*

$$\text{pr}^{(n)}v|_{(x, u^{(n)})} = \left. \frac{d}{d\varepsilon} \right|_0 \text{pr}^{(n)}(\text{Fl}_\varepsilon^v)(x, u^{(n)}),$$

for any $(x, u^{(n)}) \in M^{(n)}$.

Note that from this definition, together with 3.1.9 it immediately follows that

$$\text{pr}^{(n)}(\text{Fl}_\varepsilon^v) = \text{Fl}_\varepsilon^{\text{pr}^{(n)}v}. \quad (3.10.1)$$

Any vector field v on $M = X \times U$ can be written in the form

$$v|_{(x,u)} = \sum_{i=1}^p \xi^i(x, u) \partial_{x^i} + \sum_{\alpha=1}^q \phi_\alpha(x, u) \partial_{u^\alpha}.$$

Its n -th prolongation, being a vector field on $M^{(n)}$, therefore has to be of the form

$$\text{pr}^{(n)}v|_{(x,u^{(n)})} = \sum_{i=1}^p \xi^i(x, u) \partial_{x^i} + \sum_{\alpha=1}^q \sum_J \phi_\alpha^J(x, u^{(n)}) \partial_{u_\alpha^J}, \quad (3.10.2)$$

where the last sum is over all J with $0 \leq \#J \leq n$. Note that due to (3.8.4) the prolonged group action agrees with the original one on $M^{(0)} = M$, so the coefficients ξ^i and $\phi_\alpha^0 = \phi_\alpha$ agree with those of v . Moreover, for the same reason, if $\#J = k$ then the coefficient ϕ_α^J of $\partial_{u_\alpha^J}$ will only depend on k -th and lower order derivatives of u , $\phi_\alpha^J = \phi_\alpha^J(x, u^{(k)})$, i.e.,

$$T\pi_k^n(\text{pr}^{(n)}v) = \text{pr}^{(k)}v \quad n \geq k, \quad (3.10.3)$$

where $\text{pr}^{(0)} = v$. Again this reflects the possibility of calculating the coefficients ϕ_α^J recursively. Our main goal will be to derive a general formula for calculating the ϕ_α^J given ϕ_α .

3.10.2 Example. Returning once more to the rotation group $G = SO(2)$ on \mathbb{R}^2 from 3.5.1, 3.8.2, with infinitesimal generator $v = -u\partial_x + x\partial_u$ and group action

$$\text{Fl}_\varepsilon^v(x, u) = \varepsilon \cdot (x, u) = (x \cos \varepsilon - u \sin \varepsilon, x \sin \varepsilon + u \cos \varepsilon)$$

we have

$$\text{pr}^{(1)}[\text{Fl}_\varepsilon^v](x, u, u_x) = \left(x \cos \varepsilon - u \sin \varepsilon, x \sin \varepsilon + u \cos \varepsilon, \frac{\sin \varepsilon + u_x \cos \varepsilon}{\cos \varepsilon - u_x \sin \varepsilon} \right),$$

By 3.10.1, the first prolongation of v is calculated by differentiating this expression at $\varepsilon = 0$. This gives

$$\text{pr}^{(1)}v = -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x} \quad (3.10.4)$$

As predicted by (3.10.3), the first two terms coincide with those of v itself.

In order to be able to apply the criterion 3.2.13 to the present situation we need a suitable maximal rank condition for systems of differential equations.

3.10.3 Definition. *A system*

$$P_\nu(x, u^{(n)}) = 0 \quad \nu = 1, \dots, l$$

of differential equations is said to be of maximal rank if the $l \times (p + qp^{(n)})$ Jacobian matrix

$$J_P(x, u^{(n)}) = \left(\frac{\partial P_\nu}{\partial x^i}, \frac{\partial P_\nu}{\partial u_\alpha^J} \right)$$

of P with respect to all its variables $(x, u^{(n)})$ is of rank l whenever $P(x, u^{(n)}) = 0$.

3.10.4 Example. (i) The 2-dimensional Laplace equation $P = u_{xx} + u_{yy} = 0$ is of maximal rank since the Jacobian of P with respect to $(x, y; u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}) \in X \times U^{(2)}$ is

$$J_P = (0, 0; 0; 0, 0; 1, 0, 1),$$

which has rank 1 everywhere.

(ii) The equation $P = (u_{xx} + u_{yy})^2 = 0$ is not of maximal rank, because

$$J_P = (0, 0; 0; 0, 0; 2(u_{xx} + u_{yy}), 0, 2(u_{xx} + u_{yy}))$$

vanishes on $P = 0$.

3.10.5 Remark. In practice, the maximal rank condition is not much of a restriction, i.e., ‘most’ systems automatically satisfy it. In fact, if $\mathcal{S}_P = \{(x, u^{(n)}) \mid P(x, u^{(n)}) = 0\}$ is a regular (l -dimensional) submanifold of $M^{(n)}$ then locally around any point in $M^{(n)}$ we can pick an adapted coordinate system (cf. [10, 1.1.12]), and thereby (extracting the relevant coordinates) find a smooth map \tilde{P} of rank l such that $\mathcal{S}_P = \mathcal{S}_{\tilde{P}} = \{(x, u^{(n)}) \mid \tilde{P}(x, u^{(n)}) = 0\}$.

Using the above condition we can now formulate the main result on the calculation of symmetry groups of systems of differential equations:

3.10.6 Theorem. *Let*

$$P_\nu(x, u^{(n)}) = 0 \quad \nu = 1, \dots, l$$

be a system of differential equations of maximal rank defined on an open subset M of $X \times U$. If G is a local transformation group acting on M and

$$\text{pr}^{(n)}v[P_\nu(x, u^{(n)})] = 0, \quad \nu = 1, \dots, l, \text{ whenever } P(x, u^{(n)}) = 0 \quad (3.10.5)$$

for every infinitesimal generator v of G , then G is a symmetry group of the system.

Proof. By 3.9.1 it suffices to show that \mathcal{S}_P remains invariant under $\text{pr}^{(n)}G$. By 3.8.1, $\text{pr}^{(n)}G$ is a local group of transformations on $M^{(n)}$, whose infinitesimal generators are exactly the $\text{pr}^{(n)}v$, for v the infinitesimal generators of G (by 3.10.1). Since P is of maximal rank, the result now follows from 3.2.13. \square

We shall see in 3.12.4 that if P satisfies a certain local solvability condition then (3.10.5) is in fact necessary and sufficient for G to be a symmetry group of the system.

The Hadamard Lemma 3.2.15 allows the following equivalent reformulation of (3.10.5): there exist smooth functions $Q_{\nu\mu} = Q_{\nu\mu}(x, u^{(n)})$ such that

$$\text{pr}^{(n)}v[P_\nu(x, u^{(n)})] = \sum_{\mu=1}^l Q_{\nu\mu}(x, u^{(n)}) P_\mu(x, u^{(n)}) \quad (3.10.6)$$

holds identically on $M^{(n)}$.

3.10.7 Example. Continuing our analysis of the rotation group $SO(2)$ on \mathbb{R}^2 from 3.10.2, consider the first order ODE

$$P(x, u, u_x) = (u - x)u_x + u + x = 0. \quad (3.10.7)$$

The Jacobian of P is

$$J_P = (\partial_x P, \partial_u P, \partial_{u_x} P) = (1 - u_x, 1 + u_x, u - x),$$

hence has rank 1 everywhere.

We now apply the infinitesimal generator $\text{pr}^{(1)}v$ from (3.10.4) to P :

$$\begin{aligned}\text{pr}^{(1)}v(P) &= -u\partial_x P + x\partial_u P + (1 + u_x^2)\partial_{u_x} P \\ &= -u(1 - u_x) + x(1 + u_x) + (1 + u_x)^2(u - x) \\ &= u_x[(u - x)u_x + u + x] = u_x P.\end{aligned}$$

Therefore, (3.10.5) (or also (3.10.6)) is satisfied, and we conclude that the rotation group transforms solutions of (3.10.7) to other solutions.

The calculation of even the first prolongation of the infinitesimal generator of the rotation group in 3.8.2 demonstrates that the direct method of first determining the prolonged group action and then calculating the corresponding generators is not feasible in practice. We need to find an algorithmic way of directly calculating prolongations of vector fields, avoiding the detour of determining the prolonged group action along the way.

Before deriving a general formula, let us start out by considering some special cases first. Let

$$v = \sum_{i=1}^p \xi^i(x) \partial_{x^i}$$

on $M \subseteq X \times U$, with $U = \mathbb{R}$. According to (3.5.2), the corresponding group action $g_\varepsilon = \text{Fl}_\varepsilon^v$ then is of the form

$$(\tilde{x}, \tilde{u}) = g_\varepsilon \cdot (x, u) = (\Xi_\varepsilon(x), u),$$

where

$$\left. \frac{d}{d\varepsilon} \right|_0 \Xi_\varepsilon^i(x) = \xi^i(x). \quad (3.10.8)$$

On $M^{(1)}$ we have coordinates $(x, u^{(1)}) = (x^i, u, u_j)$ with $u_j := \partial_{x^j} u$. By definition, to calculate $\text{pr}^{(1)}v$ in $(x, u^{(1)})$, we may take any function f with $u = f(x)$ and $u_j = \partial_{x^j} f(x)$ for $j = 1, \dots, p$. Then

$$\text{pr}^{(1)}g_\varepsilon \cdot (x, u^{(1)}) = (\tilde{x}, \tilde{u}^{(1)}),$$

where $\tilde{x} = \Xi_\varepsilon(x)$, $\tilde{u} = u$, and \tilde{u}_j are the derivatives of the transformed function $\tilde{f}_\varepsilon = g_\varepsilon \cdot f$, which, by (3.5.3), is given by

$$\tilde{u} = \tilde{f}_\varepsilon(\tilde{x}) = f(\Xi_\varepsilon^{-1}(\tilde{x})) = f(\Xi_{-\varepsilon}(\tilde{x})).$$

Therefore,

$$\tilde{u}_j = \frac{\partial \tilde{f}_\varepsilon}{\partial \tilde{x}^j}(\tilde{x}) = \sum_{k=1}^p \frac{\partial f}{\partial x^k}(\Xi_{-\varepsilon}(\tilde{x})) \cdot \frac{\partial \Xi_{-\varepsilon}^k}{\partial \tilde{x}^j}(\tilde{x}). \quad (3.10.9)$$

Since $\Xi_{-\varepsilon}(\tilde{x}) = x$, this simplifies to

$$\tilde{u}_j = \sum_{k=1}^p \frac{\partial \Xi_{-\varepsilon}^k}{\partial \tilde{x}^j}(\Xi_\varepsilon(x)) u_k.$$

This is the explicit formula for the action of g_ε on the u_j -variables. To find the infinitesimal generator of $\text{pr}^{(1)}g_\varepsilon$, we have to differentiate this expression with respect to ε at $\varepsilon = 0$. Also, since $\text{pr}^{(1)}g_\varepsilon$ acts on (x, u) as g_ε , the coefficients of ∂_{x^i} and ∂_u in $\text{pr}^{(1)}v$ stay the same. Hence

$$\text{pr}^{(1)}v = \sum_{i=1}^p \xi^i(x) \partial_{x^i} + \sum_{j=1}^p \phi^j(x, u^{(1)}) \partial_{u_j}, \quad (3.10.10)$$

where

$$\phi^j(x, u^{(1)}) = \frac{d}{d\varepsilon} \Big|_0 \sum_{k=1}^p \frac{\partial \Xi_{-\varepsilon}^k}{\partial \tilde{x}^j}(\Xi_\varepsilon(x)) u_k.$$

Here, using (3.10.8), we have

$$\frac{d}{d\varepsilon} \Big|_0 \left[\frac{\partial \Xi_{-\varepsilon}^k}{\partial \tilde{x}^j}(\Xi_\varepsilon(x)) \right] = -\frac{\partial \xi^k}{\partial x^j}(x) + \sum_{l=1}^p \left[\frac{\partial^2 \Xi_{-\varepsilon}^k}{\partial \tilde{x}^j \partial \tilde{x}^l}(\Xi_\varepsilon(x)) \frac{d}{d\varepsilon} \Xi_\varepsilon^l(x) \right]_{\varepsilon=0},$$

and the second term vanishes since $\Xi_0 = \text{id}$. Therefore,

$$\phi^j(x, u, u_x) = - \sum_{k=1}^p \frac{\partial \xi^k}{\partial x^j} u_k. \quad (3.10.11)$$

The second special case we look at is a local group action as above, only this time acting exclusively on the dependent variable:

$$(\tilde{x}, \tilde{u}) = g_\varepsilon \cdot (x, u) = (x, \Phi_\varepsilon(x, u)).$$

In this case, $v = \phi(x, u) \partial_u$, where

$$\phi(x, u) = \frac{d}{d\varepsilon} \Big|_0 \Phi_\varepsilon(x, u).$$

Let f be a local smooth function with $f(x) = u$. Then $\tilde{f}_\varepsilon = g_\varepsilon \cdot f$ is given by

$$\tilde{u} = \tilde{f}_\varepsilon(x) = \Phi_\varepsilon(x, f(x)). \quad (3.10.12)$$

To determine the prolonged group action, we need to differentiate \tilde{f}_ε :

$$\tilde{u}_j = \frac{\partial \tilde{f}_\varepsilon}{\partial x^j}(x) = \frac{\partial \Phi_\varepsilon}{\partial x^j}(x, f(x)) + \frac{\partial \Phi_\varepsilon}{\partial u}(x, f(x)) \frac{\partial f}{\partial x^j}(x).$$

Consequently, $\text{pr}^{(1)} g_\varepsilon \cdot (x, u^{(1)}) = (x, \tilde{u}^{(1)})$, where

$$\tilde{u}_j = \frac{\partial \Phi_\varepsilon}{\partial x^j} + \frac{\partial \Phi_\varepsilon}{\partial u} u_j \quad (3.10.13)$$

As before, to determine ϕ^j in

$$\text{pr}^{(1)} v = v + \sum_{j=1}^p \phi^j(x, u^{(1)}) \partial_{u_j},$$

we have to differentiate (3.10.13) at $\varepsilon = 0$:

$$\phi^j(x, u^{(1)}) = \frac{d}{d\varepsilon} \Big|_0 \tilde{u}_j = \frac{\partial \phi}{\partial x^j} + u_j \frac{\partial \phi}{\partial u}. \quad (3.10.14)$$

We have here the first instance of a generally useful operation, namely that of total derivative:

$$\phi^j(x, u^{(1)}) = \frac{\partial}{\partial x^j} [\phi(x, f(x))].$$

Thus $\phi^j(x, u^{(1)})$ is calculated from $\phi(x, u)$ by differentiating with respect to x^j while treating u as a function of x . Accordingly, the total derivative of ϕ with respect to x^j , D_j is defined as

$$D_j \phi := \frac{\partial \phi}{\partial x^j} + u_j \frac{\partial \phi}{\partial u}.$$

More generally, we define:

3.10.8 Definition. Let $(x, u^{(n)}) \mapsto P(x, u^{(n)})$ be a smooth function of x , u , and derivatives of u up to order n , defined on an open subset $M^{(n)}$ of $X \times U^{(n)}$. The total derivative of P with respect to x^i is the unique smooth function $D_i P(x, u^{(n+1)})$ defined on $M^{(n+1)}$ with the property that if $u = f(x)$ is any smooth function of x then

$$D_i P(x, \text{pr}^{(n+1)} f(x)) = \frac{\partial}{\partial x^i} [P(x, \text{pr}^{(n)} f(x))].$$

Thus $D_i P$ is calculated by differentiating P with respect to x^i while treating all u^α and their derivatives as functions of x .

3.10.9 Lemma. For any $P(x, u^{(n)})$ we have

$$D_i P = \frac{\partial P}{\partial x^i} + \sum_{\alpha=1}^q \sum_J u_{J,i}^\alpha \frac{\partial P}{\partial u_J^\alpha}, \quad (3.10.15)$$

where, for $J = (j_1, \dots, j_k)$,

$$u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i} = \frac{\partial^{k+1} u^\alpha}{\partial x^i \partial x^{j_1} \dots \partial x^{j_k}}, \quad (3.10.16)$$

and the sum in (3.10.15) extends over all $0 \leq \#J \leq n$.

For example, with $X = \mathbb{R}^2$, $U = \mathbb{R}$ we have

$$\begin{aligned} D_x P &= \frac{\partial P}{\partial x} + u_x \frac{\partial P}{\partial u} + u_{xx} \frac{\partial P}{\partial u_x} + u_{xy} \frac{\partial P}{\partial u_y} + u_{xxx} \frac{\partial P}{\partial u_{xx}} + \dots \\ D_y P &= \frac{\partial P}{\partial y} + u_y \frac{\partial P}{\partial u} + u_{xy} \frac{\partial P}{\partial u_x} + u_{yy} \frac{\partial P}{\partial u_y} + u_{xyy} \frac{\partial P}{\partial u_{xx}} + \dots \end{aligned}$$

Higher order total derivatives are defined inductively: let $J = (j_1, \dots, j_k)$ be a k -th order multiindex with $1 \leq j_\kappa \leq p$ for each κ , then we set

$$D_J := D_{j_1} D_{j_2} \dots D_{j_k}.$$

Also, note that the order of differentiation does not matter for total derivatives: $D_j D_l = D_l D_j$ for all j, l .

In the proof of the general prolongation formula we will need the following auxiliary result:

3.10.10 Lemma. Let $\varepsilon \mapsto M(\varepsilon)$ be a smooth map that takes values in the space of invertible $n \times n$ -matrices. Then

$$\frac{d}{d\varepsilon} [M(\varepsilon)^{-1}] = -M(\varepsilon)^{-1} \frac{dM(\varepsilon)}{d\varepsilon} M(\varepsilon)^{-1}.$$

Proof. It suffices to differentiate the identity $M(\varepsilon)M(\varepsilon)^{-1} = I$ by the product rule. \square

3.10.11 Remark. As a final prerequisite for the proof of 3.10.12 below we need the observation that given any open subset M of $X \times U$, one can view the $(n+1)$ -st jet space $M^{(n+1)}$ as a subspace of the first jet space $(M^{(n)})^{(1)}$ of the n -th jet space. In fact, any $(n+1)$ -st derivative u_J^α can be viewed as a first order derivative of an n -th order derivative (which, in general, can be done in many different ways). For example, let $p = 2$, $q = 1$, so that the coordinates on $M^{(1)}$

can be written as $(x, y; u; u_x, u_y)$. Then we may view (u_x, u_y) as new dependent variables, $u_x = v$, $u_y = w$. This turns $M^{(1)}$ into a subset of $X \times \tilde{U}$, where X is still \mathbb{R}^2 , but \tilde{U} has three dependent variables u, v, w . Consequently, the first jet space $(M^{(1)})^{(1)}$ of $M^{(1)}$ turns into an open subset of $X \times \tilde{U}^{(1)}$, with coordinates $(x, y; u; v; w; u_x, u_y, v_x, v_y, w_x, w_y)$. Since we defined $v = u_x$, $w = u_y$ it follows that $M^{(2)} \subseteq (M^{(1)})^{(1)}$ is the subspace defined by the relations

$$v = u_x, \quad w = u_y, \quad v_y = w_x$$

in $X \times \tilde{U}^{(1)}$. Here, the third relation is a consequence of $u_{xy} = u_{yx}$.

3.10.12 Theorem. *(The general prolongation formula) Let*

$$v = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha} \quad (3.10.17)$$

be a vector field on an open subset $M \subseteq X \times U$. The n -th prolongation of v is the vector field

$$\text{pr}^{(n)}v = v + \sum_{\alpha=1}^q \sum_J \phi_\alpha^J(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha} \quad (3.10.18)$$

defined on $M^{(n)} \subseteq X \times U^{(n)}$, where J runs over all multi-indices $J = (j_1, \dots, j_k)$ with $1 \leq j_k \leq p$, $1 \leq k \leq n$. The coefficient functions ϕ_α^J are given by

$$\phi_\alpha^J(x, u^{(n)}) = D_J \left(\phi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha \right) + \sum_{i=1}^p \xi^i u_{J,i}^\alpha, \quad (3.10.19)$$

where $u_i^\alpha = \partial_{x^i} u^\alpha$ and $u_{J,i}^\alpha = \partial_{x^i} u_J^\alpha$.

Proof. We start out with the case $n = 1$. Let $g_\varepsilon = \text{Fl}_\varepsilon^v$ and

$$(\tilde{x}, \tilde{u}) = g_\varepsilon \cdot (x, u) = (\Xi_\varepsilon(x, u), \Phi_\varepsilon(x, u)).$$

Then

$$\begin{aligned} \xi^i(x, u) &= \left. \frac{d}{d\varepsilon} \right|_0 \Xi_\varepsilon^i(x, u), \quad i = 1, \dots, p, \\ \phi_\alpha(x, u) &= \left. \frac{d}{d\varepsilon} \right|_0 \Phi_\varepsilon^\alpha(x, u), \quad \alpha = 1, \dots, q. \end{aligned} \quad (3.10.20)$$

Given any $(x, u^{(1)}) \in M^{(1)}$, pick a local smooth map f with $\text{pr}^{(1)}f(x) = u^{(1)}$, i.e., with $u^\alpha = f^\alpha(x)$, $u_i^\alpha = \partial_{x^i} f^\alpha(x)$. By (3.5.3), for ε small and restricting the domain of f sufficiently, we have

$$\tilde{u} = \tilde{f}_\varepsilon(\tilde{x}) = (g_\varepsilon \cdot f)(\tilde{x}) = [\Phi_\varepsilon \circ (\text{id} \times f)] \circ [\Xi_\varepsilon \circ (\text{id} \times f)]^{-1}(\tilde{x}).$$

By the chain rule, we conclude from this that for the Jacobian $J\tilde{f}_\varepsilon = (\partial \tilde{f}_\varepsilon^\alpha / \partial \tilde{x}^i)$ we have (using that $x = [\Xi_\varepsilon \circ (\text{id} \times f)]^{-1}(\tilde{x})$)

$$J\tilde{f}_\varepsilon(\tilde{x}) = J[\Phi_\varepsilon \circ (\text{id} \times f)](x) \cdot [J[\Xi_\varepsilon \circ (\text{id} \times f)](x)]^{-1} \quad (3.10.21)$$

The matrix entries of $J\tilde{f}_\varepsilon(\tilde{x})$ then give formulae for $\text{pr}^{(1)}g_\varepsilon$.

To calculate $\text{pr}^{(1)}v$ we have to differentiate (3.10.21) at $\varepsilon = 0$. In doing this, note that since $g_0 = \text{id}$, we have

$$\Xi_0(x, f(x)) = x, \quad \Phi_0(x, f(x)) = f(x), \quad (3.10.22)$$

so with I the $p \times p$ unit matrix we get

$$J[\Xi_0 \circ (\text{id} \times f)](x) = I, \quad J[\Phi_0 \circ (\text{id} \times f)](x) = Jf(x).$$

Therefore, using 3.10.10,

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_0 J\tilde{f}_\varepsilon(\tilde{x}) &= \frac{d}{d\varepsilon} \Big|_0 J[\Phi_\varepsilon \circ (\text{id} \times f)](x) - Jf(x) \frac{d}{d\varepsilon} \Big|_0 J[\Xi_\varepsilon \circ (\text{id} \times f)](x) \\ &= J[\phi \circ (\text{id} \times f)](x) - Jf(x) \cdot J[\xi \circ (\text{id} \times f)](x). \end{aligned}$$

The matrix entries of this expression give the coefficient functions ϕ_α^k of $\partial_{u_k^\alpha}$ in $\text{pr}^{(1)}v$. The (α, k) -th entry is

$$\phi_\alpha^k(x, \text{pr}^{(1)}f(x)) = \frac{\partial}{\partial x^k} [\phi_\alpha(x, f(x))] - \sum_{i=1}^p \frac{\partial f^\alpha}{\partial x^i} \cdot \frac{\partial}{\partial x^k} [\xi^i(x, f(x))],$$

or, in terms of total derivatives,

$$\begin{aligned} \phi_\alpha^k(x, u^{(1)}) &= D_k \phi_\alpha(x, u^{(1)}) - \sum_{i=1}^p D_k \xi^i(x, u^{(1)}) u_i^\alpha \\ &= D_k \left[\phi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha \right] + \sum_{i=1}^p \xi^i u_{ki}^\alpha, \end{aligned} \tag{3.10.23}$$

with $u_{ki}^\alpha = \partial^2 u^\alpha / \partial x^k \partial x^i$. This is (3.10.19) for $n = 1$.

In the general case we proceed by induction. Here we make use of the key observation 3.10.11, which allows us to view the $(n+1)$ -st jet space $M^{(n+1)}$ as a subspace of the first jet space $(M^{(n)})^{(1)}$ of the n -th jet space. The strategy is to regard $\text{pr}^{(n-1)}v$ as a vector field $M^{(n-1)}$ and then use the case $n = 1$ above to prolong it to $(M^{(n-1)})^{(1)}$. Then we restrict the resulting vector field to the subspace $M^{(n)}$ of $(M^{(n-1)})^{(1)}$, and this will give the n -th prolongation of v . The fact that this is possible and gives a well-defined result will follow from the explicit formula below.

The coordinates on $(M^{(n-1)})^{(1)}$ are given by $u_{J,k}^\alpha = \frac{\partial u_J^\alpha}{\partial x^k}$, where $J = (j_1, \dots, j_{n-1})$, $1 \leq k \leq p$, and $1 \leq \alpha \leq q$. By (3.10.3), the only new coefficients we need to calculate are those of the highest order derivatives $\partial / \partial u_{J,k}^\alpha$. By (3.10.23), they are given by

$$\phi_\alpha^{J,k} = D_k \phi_\alpha^J - \sum_{i=1}^p D_k \xi^i \cdot u_{J,i}^\alpha. \tag{3.10.24}$$

To finish the proof, it suffices to show that the explicit formula (3.10.19) solves the recursion relation (3.10.24) in closed form. This follows by induction: supposing that ϕ_α^J has already been shown to satisfy (3.10.19), it follows from (3.10.24) that

$$\begin{aligned} \phi_\alpha^{J,k} &= D_k \left[D_J \left(\phi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha \right) + \sum_{i=1}^p \xi^i u_{J,i}^\alpha \right] - \sum_{i=1}^p D_k \xi^i \cdot u_{J,i}^\alpha \\ &= D_k D_J \left(\phi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha \right) + \sum_{i=1}^p (D_k \xi^i \cdot u_{J,i}^\alpha + \xi^i u_{J,ik}^\alpha) - \sum_{i=1}^p D_k \xi^i \cdot u_{J,i}^\alpha \\ &= D_k D_J \left(\phi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha \right) + \sum_{i=1}^p \xi^i u_{J,ik}^\alpha, \end{aligned}$$

where $u_{J,ik}^\alpha = \partial^2 u_J^\alpha / \partial x^i \partial x^k$. Thus $\phi_\alpha^{J,k}$ is of the form (3.10.19) as well, which completes the induction step. \square

3.10.13 Remark. Note that (3.10.24) provides a useful way of calculating prolongations of vector fields recursively.

3.10.14 Example. As an illustration of the relative ease with which one may calculate prolongations of vector fields using the general prolongation formula, we once more return to the rotation group $G = SO(2)$ acting on \mathbb{R}^2 , cf. 3.10.7. The infinitesimal generator is $v = -u\partial_x + x\partial_u$, so $\phi = x$ and $\xi = -u$. By 3.10.12 we have

$$\text{pr}^{(1)}v = v + \phi^x \partial_{u_x},$$

where

$$\phi^x = D_x(\phi - \xi u_x) + \xi u_{xx} = D_x(x + uu_x) - uu_{xx} = 1 + u_x^2,$$

which confirms (3.10.4). To calculate the coefficient ϕ^{xx} of $\partial_{u_{xx}}$ in $\text{pr}^{(2)}v$, we can either use (3.10.19):

$$\phi^{xx} = D_x^2(\phi - \xi u_x) + \xi u_{xxx} = D_x^2(x + uu_x) - uu_{xxx} = 3u_x u_{xx},$$

or the recursion formula (3.10.24):

$$\phi^{xx} = D_x \phi^x - u_{xx} D_x \xi = D_x(1 + u_x^2) + u_x u_{xx} = 3u_x u_{xx}.$$

Consequently,

$$\text{pr}^{(2)}v = -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x} + 3u_x u_{xx} \partial_{u_{xx}}.$$

Note that to derive this result via the second prolongation of the group action (as in 3.8.2) would be much more involved.

Based on this, and using the invariance criterion in 3.10.6, it follows that the ODE $u_{xx} = 0$ has $SO(2)$ as a symmetry group. In fact,

$$\text{pr}^{(2)}v(u_{xx}) = 3u_x u_{xx} = 0 \quad \text{whenever} \quad u_{xx} = 0.$$

Since the solutions of $u_{xx} = 0$ are linear functions, this is basically just the statement that rotations take straight lines to straight lines.

Next, consider the function

$$\kappa(x, u^{(2)}) = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}, \quad (3.10.25)$$

describing the Frenet-curvature of a planar curve. Then one easily checks that $\text{pr}^{(2)}v(\kappa) = 0$, so 3.2.7 shows that κ is an invariant of $\text{pr}^{(2)}SO(2)$. Geometrically, this means that the curvature of a planar curve is invariant under rotations.

3.10.15 Theorem. Let v, w be smooth vector fields on the open subset M of $X \times U$. Then

$$(i) \quad \text{pr}^{(n)}(cv + w) = c \cdot \text{pr}^{(n)}v + \text{pr}^{(n)}w \quad (c \in \mathbb{R}).$$

$$(ii) \quad \text{pr}^{(n)}[v, w] = [\text{pr}^{(n)}v, \text{pr}^{(n)}w].$$

Proof. (i) This is immediate from 3.10.12.

(ii) This also could be verified using 3.10.12, but we take a different route. First, by 3.8.1 we have

$$\text{pr}^{(n)}(g \cdot h) = (\text{pr}^{(n)}g) \cdot (\text{pr}^{(n)}h).$$

Also, defining $(g + h) \cdot x := g \cdot x + h \cdot x$ we obtain

$$\text{pr}^{(n)}(g + h) = \text{pr}^{(n)}g + \text{pr}^{(n)}h$$

directly from the definition (3.8.2) (using sums of representing functions to represent sums of points in the jet bundle). Also, $\text{pr}^{(n)}\text{id}_M = \text{id}_{M^{(n)}}$, and $v \mapsto \text{pr}^{(n)}v$ is clearly continuous. Finally, recall from (3.10.1) that

$$\text{pr}^{(n)}(\text{Fl}_\varepsilon^v) = \text{Fl}_\varepsilon^{\text{pr}^{(n)}v},$$

so we can use 2.1.20 to calculate as follows:

$$\begin{aligned} [\text{pr}^{(n)}v, \text{pr}^{(n)}w] &= \lim_{\varepsilon \rightarrow 0} \frac{\text{Fl}_{-\sqrt{\varepsilon}}^{\text{pr}^{(n)}w} \circ \text{Fl}_{-\sqrt{\varepsilon}}^{\text{pr}^{(n)}v} \circ \text{Fl}_{\sqrt{\varepsilon}}^{\text{pr}^{(n)}w} \circ \text{Fl}_{\sqrt{\varepsilon}}^{\text{pr}^{(n)}v} - \text{id}_{M^{(n)}}}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\text{pr}^{(n)}[\text{Fl}_{-\sqrt{\varepsilon}}^w \circ \text{Fl}_{-\sqrt{\varepsilon}}^v \circ \text{Fl}_{\sqrt{\varepsilon}}^w \circ \text{Fl}_{\sqrt{\varepsilon}}^v - \text{id}_M]}{\varepsilon} \\ &= \text{pr}^{(n)} \left[\lim_{\varepsilon \rightarrow 0} \frac{\text{Fl}_{-\sqrt{\varepsilon}}^w \circ \text{Fl}_{-\sqrt{\varepsilon}}^v \circ \text{Fl}_{\sqrt{\varepsilon}}^w \circ \text{Fl}_{\sqrt{\varepsilon}}^v - \text{id}_M}{\varepsilon} \right] \\ &= \text{pr}^{(n)}[v, w]. \end{aligned}$$

□

If we call a vector field v an *infinitesimal symmetry* of a system \mathcal{S} of differential equations if it satisfies (3.10.5), then we obtain from the previous theorem:

3.10.16 Corollary. *Let \mathcal{S} be a system of differential equations of maximal rank defined on $M \subseteq X \times U$. Then the set of all infinitesimal symmetries of \mathcal{S} forms a Lie algebra of vector fields on M .*

To conclude this section we note the following equivalent way of writing the general prolongation formula 3.10.12. For v a vector field on $M \subseteq X \times U$, let

$$Q_\alpha(x, u^{(1)}) := \phi_\alpha(x, u) - \sum_{i=1}^p \xi^i(x, u) u_i^\alpha, \quad \alpha = 1, \dots, q. \quad (3.10.26)$$

The q -tuple $Q(x, u^{(1)}) = (Q_1, \dots, Q_q)$ is called the *characteristic* of the vector field v . Using this, (3.10.19) becomes

$$\phi_\alpha^J = D_J Q_\alpha + \sum_{i=1}^p \xi^i u_{J,i}^\alpha \quad (3.10.27)$$

Substituting this into (3.10.18), a brief calculation gives

$$\text{pr}^{(n)}v = \sum_{\alpha=1}^q \sum_J D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha} + \sum_{i=1}^p \xi^i \left(\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \sum_J u_{J,i}^\alpha \frac{\partial}{\partial u_J^\alpha} \right), \quad (3.10.28)$$

which, by (3.10.15), means that

$$\text{pr}^{(n)}v = \text{pr}^{(n)}v_Q + \sum_{i=1}^p \xi^i D_i, \quad (3.10.29)$$

where

$$v_Q := \sum_{\alpha=1}^q Q_\alpha(x, u^{(1)}) \frac{\partial}{\partial u^\alpha}, \quad \text{pr}^{(n)}v_Q = \sum_{\alpha=1}^q \sum_J D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha}. \quad (3.10.30)$$

Here, the last equality comes from applying (3.10.28) to v_Q . In all of the above equations, the sums over J extend over all J with $0 \leq \#J \leq n$. Note that the individual terms on the right hand side of (3.10.29) actually involve $(n+1)$ -st order derivatives of u , and only their combination gives a genuine vector field on $M^{(n)}$.

3.11 Calculation of symmetry groups

In this section we want to illustrate how to explicitly calculate the symmetry group of any given system \mathcal{S} of differential equations. From the previous sections, we obtain the following general strategy for tackling this problem:

- (i) Make an ansatz for an infinitesimal generator v of a prospective one-parameter symmetry group of \mathcal{S} in the form (3.10.17) with unknown functions ξ_i and ϕ_α .
- (ii) Calculate $\text{pr}^{(n)}v$ according to (3.10.18), (3.10.19), where n is the order of \mathcal{S} .
- (iii) Insert the resulting expression into (3.10.5). Since these equations only have to hold on \mathcal{S}_P , eliminate any dependencies of the derivatives of the u 's by using the equations in \mathcal{S} . The equations achieved in this way have to hold identically in x, u and the remaining partial derivatives of the u 's.
- (iv) Solving the resulting so-called *defining equations* produces a number of (usually elementary) partial differential equations for the unknown functions ξ_i and ϕ_α .
- (v) Compute the ξ_i and ϕ_α from these equations. By 3.10.16, the vector fields gathered from this procedure form a Lie algebra of infinitesimal symmetries.
- (vi) The corresponding one-parameter symmetry groups are found by calculating the flows of the respective infinitesimal symmetries.

Let us systematically work through this list in a concrete example:

3.11.1 Example. Consider the one-dimensional heat equation

$$u_t = u_{xx} \quad (3.11.1)$$

Thus we have $X = \mathbb{R}^2$ with variables x, t , and $U = \mathbb{R}$, and the equation $P(x, t, u^{(2)}) = u_t - u_{xx}$ determines a subvariety in $X \times U^{(2)}$. For the infinitesimal generators of symmetries of $\mathcal{S} = \{P = 0\}$ we make the ansatz

$$v = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \phi(x, t, u) \frac{\partial}{\partial u} \quad (3.11.2)$$

Our aim is to determine all possible coefficient functions ξ, τ, ϕ such that Fl_ε^v is a symmetry group of the heat equation (3.11.1). Since we wish to apply 3.10.6, we will need the second prolongation of v ,

$$\text{pr}^{(2)}v = v + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}.$$

Inserting this into (3.10.5), it follows that the only restriction on the coefficients of v is that

$$\phi^t = \phi^{xx} \quad \text{whenever} \quad u_t = u_{xx}. \quad (3.11.3)$$

Using (3.10.19), we calculate

$$\begin{aligned} \phi^t &= D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt} = D_t\phi - u_x D_t\xi - u_t D_t\tau \\ &= \phi_t - \xi_t u_x + (\phi_u - \tau_t) u_t - \xi_u u_x u_t - \tau_u u_t^2 \end{aligned} \quad (3.11.4)$$

and

$$\begin{aligned}
\phi^{xx} &= D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt} \\
&= D_x^2\phi - u_x D_x^2\xi - u_t D_x^2\tau - 2u_{xx} D_x\xi - 2u_{xt} D_x\tau \\
&= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_xu_t \\
&\quad - \xi_{uu}u_x^3 - \tau_{uu}u_x^2u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_xu_{xt} - 3\xi_uu_xu_{xx} \\
&\quad - \tau_uu_tu_{xx} - 2\tau_uu_xu_{xt}.
\end{aligned} \tag{3.11.5}$$

If we insert these expressions into (3.11.3) and replace u_t by u_{xx} wherever it occurs, we obtain an equation on the jet space $X \times U^{(2)}$ that has to hold identically in all the remaining variables. We may therefore ‘split’ the equation by equating the coefficient functions of all monomials appearing in it. This gives the following table:

Monomial	Coefficient	
u_xu_{xt}	$0 = -2\tau_u$	(a)
u_{xt}	$0 = -2\tau_x$	(b)
u_{xx}^2	$-\tau_u = -\tau_u$	(c)
$u_x^2u_{xx}$	$0 = -\tau_{uu}$	(d)
u_xu_{xxx}	$-\xi_u = -2\tau_{xu} - 3\xi_u$	(e)
u_{xx}	$\phi_u - \tau_t = -\tau_{xx} + \phi_u - 2\xi_x$	(f)
u_x^3	$0 = -\xi_{uu}$	(g)
u_x^2	$0 = \phi_{uu} - 2\xi_{xu}$	(h)
u_x	$-\xi_t = 2\phi_{xu} - \xi_{xx}$	(j)
1	$\phi_t = \phi_{xx}$	(k)

We first note that (a), (b) imply that τ is a function of t only: $\tau = \tau(t)$. Using this, (e) shows that ξ does not depend on u , and (f) gives $\tau_t = 2\xi_x$. Therefore,

$$\xi(x, t) = \frac{1}{2}\tau_t x + \sigma(t),$$

where σ is some function of t . (h) shows that ϕ is linear in u , so

$$\phi(x, t, u) = \beta(x, t)u + \alpha(x, t)$$

for some functions α, β . By (j), $-2\beta_x = \xi_t = \frac{1}{2}\tau_{tt}x + \sigma_t$, so

$$\beta(x, t) = -\frac{1}{8}\tau_{tt}x^2 - \frac{1}{2}\sigma_t x + \rho(t). \tag{3.11.6}$$

Finally, (k) implies that both α and β must be solutions of the heat equation:

$$\alpha_t = \alpha_{xx}, \quad \beta_t = \beta_{xx}.$$

Combined with (3.11.6) we get

$$\tau_{ttt} = 0, \quad \sigma_{tt} = 0, \quad \rho_t = -\frac{1}{4}\tau_{tt}.$$

Hence τ is quadratic in t , σ is linear in t , so for suitable constants c_i we get:

$$\begin{aligned}
\tau &= c_2 + 2c_4t + 4c_6t^2 \\
\sigma &= c_1 + 2c_5t \\
\rho_t &= -\frac{1}{4}\tau_{tt} = -2c_6 \Rightarrow \rho = -2c_6t + c_3 \\
\Rightarrow \beta &= -\frac{1}{8}\tau_{tt}x^2 - \frac{1}{2}\sigma_t x + \rho = -c_6x^2 - c_5x - 2c_6t + c_3
\end{aligned}$$

Consequently,

$$\begin{aligned}\xi &= \frac{1}{2}\tau_t + \sigma = c_1 + c_4x + 2c_5t + 4c_6xt \\ \tau &= c_2 + 2c_4t + 4c_6t^2 \\ \phi &= \beta u + \alpha = (c_3 - c_5x - 2c_6t - c_6x^2)u + \alpha(x, t),\end{aligned}$$

where c_1, \dots, c_6 are arbitrary constants and α is any solution of the heat equation. It follows that the Lie algebra of infinitesimal symmetries of the heat equation is spanned by the six vector fields

$$\begin{aligned}v_1 &= \partial_x \\ v_2 &= \partial_t \\ v_3 &= u\partial_u \\ v_4 &= x\partial_x + 2t\partial_t \\ v_5 &= 2t\partial_x - xu\partial_u \\ v_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u\end{aligned}$$

and the infinite-dimensional subalgebra

$$v_\alpha = \alpha(x, t)\partial_u,$$

where α is any solution of the heat equation. By 3.10.16 we know that these generators span a Lie algebra of vector fields. Thus with any two generators also their Lie bracket is an infinitesimal symmetry of the heat equation.

The one-parameter symmetry group actions $\text{Fl}_\varepsilon^{v_i}$ corresponding to the generators can be calculated by solving the ODE

$$\frac{d}{d\varepsilon}(x(\varepsilon), t(\varepsilon), u(\varepsilon)) = v_i(x(\varepsilon), t(\varepsilon), u(\varepsilon))$$

with initial value $(x(0), t(0), u(0)) = (x, t, u)$. This gives the groups

$$\begin{aligned}G_1 &: (x + \varepsilon, t, u) \\ G_2 &: (x, t + \varepsilon, u) \\ G_3 &: (x, t, e^\varepsilon u) \\ G_4 &: (e^\varepsilon x, e^{2\varepsilon} t, u) \\ G_5 &: (x + 2\varepsilon t, t, ue^{-\varepsilon x - \varepsilon^2 t}) \\ G_6 &: \left(\frac{x}{1 - 4\varepsilon t}, \frac{t}{1 - 4\varepsilon t}, u\sqrt{1 - 4\varepsilon t}e^{\frac{-\varepsilon x^2}{1 - 4\varepsilon t}} \right) \\ G_\alpha &: (x, t, u + \varepsilon\alpha(x, t))\end{aligned}$$

Using (3.5.3) we conclude that, given any solution f of the heat equation, so are the following functions (given by $g^{(i)} \cdot f$, for $g^{(i)} \in G_i$):

$$\begin{aligned}u^{(1)} &= f(x - \varepsilon, t) \\ u^{(2)} &= f(x, t - \varepsilon) \\ u^{(3)} &= e^\varepsilon f(x, t) \\ u^{(4)} &= f(e^{-\varepsilon}x, e^{-2\varepsilon}t) \\ u^{(5)} &= e^{-\varepsilon x + \varepsilon^2 t} f(x - 2\varepsilon t, t) \\ u^{(6)} &= \frac{1}{\sqrt{1 + 4\varepsilon t}} e^{\frac{-\varepsilon x^2}{1 + 4\varepsilon t}} f\left(\frac{x}{1 + 4\varepsilon t}, \frac{t}{1 + 4\varepsilon t}\right) \\ u^{(\alpha)} &= f(x, t) + \varepsilon\alpha(x, t).\end{aligned}$$

Here, ε is a real number and α is any solution of the heat equation. G_3 and G_α are simply an expression of the linearity of the heat equation. G_1 and G_2 reflect the space- and time-invariance of the equation. G_4 is a scaling symmetry, and G_5 represents a kind of Galilean boost to a moving coordinate frame. G_6 is a genuinely local transformation group. Applying G_6 to a trivial solution $u = c$ (c a constant) we obtain that also

$$u = \frac{c}{\sqrt{1+4\varepsilon t}} e^{\frac{-\varepsilon x^2}{1+4\varepsilon t}}$$

is a solution. Setting $c = \sqrt{\varepsilon/\pi}$ we obtain the fundamental solution to the heat equation at the point $(x_0, t_0) = (0, -1/(4\varepsilon))$. Translating in t (using G_2) we arrive at the fundamental solution at $(0, 0)$,

$$u = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}.$$

The most general one-parameter group of symmetries has a generator of the form $c_1 v_1 + \dots + c_6 v_6 + v_\alpha$. Also, an arbitrary element of the symmetry group of the heat equation (if sufficiently close to the identity element) can be expressed in the form

$$g = \text{Fl}_{\varepsilon_\alpha}^{v_\alpha} \circ \text{Fl}_{\varepsilon_6}^{v_6} \circ \dots \circ \text{Fl}_{\varepsilon_1}^{v_1},$$

and so the most general solution obtainable from any given solution u by such a transformation is of the form

$$u = \frac{1}{\sqrt{1+4\varepsilon_6 t}} e^{\varepsilon_3 - \frac{\varepsilon_5 x + \varepsilon_6 x^2 - \varepsilon_5^2 t}{1+4\varepsilon_6 t}} f\left(\frac{e^{-\varepsilon_4}(x - 2\varepsilon_5 t)}{1+4\varepsilon_6 t} - \varepsilon_1, \frac{e^{-2\varepsilon_4 t}}{1+4\varepsilon_6 t} - \varepsilon_2\right) + \alpha(x, t),$$

with $\varepsilon_1, \dots, \varepsilon_6$ real constants and α any solution of the heat equation.

3.12 Nondegeneracy conditions

The central result 3.10.6 only gives a sufficient condition for a local group of transformations to be a symmetry group of a given system of differential equations. In the present section we will see that under some mild condition on the system the infinitesimal criterion from 3.10.6 is in fact also sufficient.

Let us first look at the difference between the criterion 3.2.13 for the invariance of a system of algebraic equations, which is necessary and sufficient, and 3.10.6. For a system of algebraic equations $F = 0$, for each point x_0 on the subvariety $\mathcal{S}_F = \{x \mid F(x) = 0\}$ there trivially exists a solution to the system, namely x_0 itself. In the case of a system $P(x, u^{(n)}) = 0$ of differential equations, however, if $(x_0, u_0^{(n)}) \in \mathcal{S}_P = \{(x, u^{(n)}) \mid P(x, u^{(n)}) = 0\}$ there is no guarantee that there exists a solution $u = f(x)$ of the differential equation locally around x_0 such that $u_0^{(n)} = \text{pr}^{(n)} f(x_0)$. Our definition 3.5.3 of a symmetry group of $P(x, u^{(n)}) = 0$ requires $\text{pr}^{(n)} G$ to transform solutions into solutions, so we only can infer that it will move points in \mathcal{S}_P for which there exists a solution as above to other points in \mathcal{S}_P . In general, these points need not constitute the entire set \mathcal{S}_P . Thus we define:

3.12.1 Definition. A system $P(x, u^{(n)}) = 0$ of n -th order differential equations is called *locally solvable at the point*

$$(x_0, u_0^{(n)}) \in \mathcal{S}_P = \{(x, u^{(n)}) \mid P(x, u^{(n)}) = 0\}$$

*if there exists a smooth function $u = f(x)$ defined near x_0 such that $u_0^{(n)} = \text{pr}^{(n)} f(x_0)$. The system is called *locally solvable* if it is locally solvable at every point of \mathcal{S}_P . It*

is called nondegenerate if it is locally solvable and of maximal rank at every point $(x_0, u_0^{(n)}) \in \mathcal{S}_P$.

3.12.2 Remark. Consider a non-singular ODE of order n , i.e., of the form

$$u_n = P(x, u, u_x, \dots, u_{n-1}), \quad (3.12.1)$$

with $u_k = \frac{\partial^k u}{\partial x^k}$ for any k and let $(x^0, u^0, \dots, u_n^0) \in \mathcal{S}_P$. Then we need to find a local smooth function $u = f(x)$ such that

$$u^0 = f(x^0), \quad u_x^0 = f'(x^0), \dots, u_{n-1}^0 = f^{(n-1)}(x^0), \quad u_n^0 = f^{(n)}(x^0).$$

Here, the first n conditions are the usual initial conditions for the ODE (3.12.1), hence can always be satisfied by standard ODE solution theory. The last condition then simply follows from (3.12.1) itself (i.e., from the fact that $(x^0, u^0, \dots, u_n^0) \in \mathcal{S}_P$). We conclude that non-singular (systems of) ODEs are always locally solvable, hence are in fact nondegenerate.

Note that even for PDEs, local solvability is usually a very mild restriction – e.g., in Cauchy problems one usually is looking for solutions with prescribed values on an entire hypersurface.

3.12.3 Example. (i) The 2-dimensional wave equation

$$P(x, t, u^{(2)}) = u_{tt} - u_{xx} = 0$$

is locally solvable. In fact, let

$$(x^0, t^0; u^0; u_x^0, u_t^0; u_{xx}^0, u_{xt}^0, u_{tt}^0) \in \mathcal{S}_P,$$

i.e., $u_{tt}^0 = u_{xx}^0$. Then we need to find a solution $u = f(x)$ of the wave equation near (x^0, t^0) with $\text{pr}^{(2)}f(x^0, t^0) = (u_x^0, u_t^0; u_{xx}^0, u_{xt}^0, u_{tt}^0)$. Since $u_{tt}^0 = u_{xx}^0$, we can take for f the polynomial solution

$$\begin{aligned} f(x, t) = & u^0 + u_x^0(x - x^0) + u_t^0(t - t^0) + \frac{1}{2}u_{xx}^0[(x - x^0)^2 + (t - t^0)^2] \\ & + u_{xt}^0(x - x^0)(t - t^0). \end{aligned}$$

(ii) The over-determined system

$$u_x = yu, \quad u_y = 0 \quad (3.12.2)$$

is not locally solvable: given (x_0, y_0) , let

$$u^0 = u(x_0, y_0) = 1, \quad u_x^0 = u_x(x_0, y_0) = y_0, \quad u_y^0 = u_y(x_0, y_0) = 0.$$

Then there is no local solution with the prescribed jet at (x_0, y_0) . In fact, by cross-differentiation we obtain

$$0 = u_{xy} = (yu)_y = yu_y + u = u,$$

which is incompatible with the above prescription.

(iii) Apart from integrability conditions as in (ii), the second main reason why a PDE may fail to be locally solvable is that there are no smooth solutions, even locally. The most famous example is due to H. Lewy ([12]), who showed that there exist smooth functions $h = h(x, y, z)$ such that the first order system

$$\begin{aligned} u_x - v_y + 2yu_z + 2xv_z &= h(x, y, z) \\ u_y + v_x - 2xu_z + 2yv_z &= 0 \end{aligned}$$

has no smooth solutions even locally around any point.

(iv) For analytic systems of PDEs (in Kovalevskaya-form), the Cauchy–Kovalevskaya theorem ensures local solvability.

For nondegenerate systems of differential equations we can now show that indeed (3.10.5) is necessary and sufficient:

3.12.4 Theorem. *Let*

$$P_\nu(x, u^{(n)}) = 0 \quad \nu = 1, \dots, l$$

be a nondegenerate system of differential equations defined on an open subset M of $X \times U$. Let G be a local transformation group acting on M . Then G is a symmetry group of the system if and only if

$$\text{pr}^{(n)}v[P_\nu(x, u^{(n)})] = 0, \quad \nu = 1, \dots, l, \text{ whenever } P(x, u^{(n)}) = 0 \quad (3.12.3)$$

for every infinitesimal generator v of G .

Proof. The condition is sufficient by 3.10.6. To see that it is also necessary, note that by 3.2.13, condition (3.12.3) is equivalent to $\text{pr}^{(n)}G$ mapping $\mathcal{S}_P = \{(x, u^{(n)}) \mid P(x, u^{(n)}) = 0\}$ into itself. Thus we have to verify that any symmetry group G of the system has this property. Let $(x_0, u_0^{(n)}) \in \mathcal{S}_P$. Then by local solvability we may find a smooth function $u = f(x)$ defined near x_0 such that $u_0^{(n)} = \text{pr}^{(n)}f(x_0)$. If $g \in G$ is such that $\text{pr}^{(n)}g \cdot (x_0, u_0^{(n)})$ is defined, then by shrinking the domain of f if necessary, we can obtain that $\tilde{f} = g \cdot f$ is a well-defined function near \tilde{x}_0 , where $(\tilde{x}_0, \tilde{u}_0) = g \cdot (x_0, u_0)$. Then by (3.8.2) we have

$$\text{pr}^{(n)}g \cdot (x_0, u_0^{(n)}) = (\tilde{x}_0, \text{pr}^{(n)}(g \cdot f)(\tilde{x}_0)) =: (\tilde{x}_0, \tilde{u}_0^{(n)}).$$

Since $\tilde{f} = g \cdot f$ is a solution, this shows that $(\tilde{x}_0, \tilde{u}_0^{(n)}) \in \mathcal{S}_P$, as claimed. \square

3.13 Integration of ordinary differential equations

As an illustration of the power of the methods developed in this chapter, in the present section we investigate the general problem of solving ordinary differential equations (ODEs). It turns out that all the standard solution techniques for special types of ODEs are instances of symmetry methods. Moreover, knowledge of a sufficiently large group of symmetries of any given ODE allows to reduce the solution procedure to successive integrations (or quadratures, in classical terminology).

Consider a first-order ODE

$$\frac{\partial u}{\partial x} = F(x, u). \quad (3.13.1)$$

We will show that if a one-parameter symmetry group of (3.13.1) is known, then it can be solved by integration. Let G be a local one-parameter group of transformations acting on an open subset M of $X \times U = \mathbb{R}^2$, with infinitesimal generator

$$v = \xi(x, u) \frac{\partial}{\partial x} + \phi(x, u) \frac{\partial}{\partial u}. \quad (3.13.2)$$

By 3.10.12, its first prolongation is given by

$$\text{pr}^{(1)}v = \xi(x, u) \frac{\partial}{\partial x} + \phi(x, u) \frac{\partial}{\partial u} + \phi^x(x, u, u_x) \frac{\partial}{\partial u_x}, \quad (3.13.3)$$

where

$$\phi^x = D_x \phi - u_x D_x \xi = \phi_x + (\phi_u - \xi_x) u_x - \xi_u u_x^2. \quad (3.13.4)$$

The infinitesimal criterion (3.10.5) for v to generate a symmetry of (3.13.1) then reads

$$\frac{\partial \phi}{\partial x} + \left(\frac{\partial \phi}{\partial u} - \frac{\partial \xi}{\partial x} \right) F - \frac{\partial \xi}{\partial u} F^2 = \xi \frac{\partial F}{\partial x} + \phi \frac{\partial F}{\partial u}, \quad (3.13.5)$$

for all (x, u) . Thus any solution to the PDE (3.13.5) generates a symmetry of the ODE (3.13.1). Although at first sight, replacing the ODE (3.13.1) by the PDE (3.13.5) may not seem to facilitate the problem, in practice one can often find symmetries directly, e.g. by geometric considerations.

Suppose that v generates a symmetry of (3.13.1) and let $v|_{(x_0, u_0)} \neq 0$. Then by straightening out (see [11, 17.14]) we may pick new coordinates

$$y = \eta(x, u), \quad w = \zeta(x, u) \quad (3.13.6)$$

near (x_0, u_0) such that in the new coordinates we have $v = \partial_w$, and thereby

$$\text{pr}^{(1)}v = v = \frac{\partial}{\partial w}$$

The condition that (3.13.1) be invariant under v then simply means that it must be independent of w , i.e., it can be written in the form

$$\frac{\partial w}{\partial y} = H(y)$$

for some smooth function H . This equation can then be solved trivially by integration:

$$w = \int H(y) dy + \text{const.}$$

Transforming back to the (x, u) -coordinates gives the solution to (3.13.1).

To find the required coordinate transform, note that in the new coordinates we require (cf. [9, 2.5.3]) that

$$v = v(\eta) \frac{\partial}{\partial y} + v(\zeta) \frac{\partial}{\partial w} \stackrel{!}{=} \frac{\partial}{\partial w},$$

so in terms of (x, u) , we need to satisfy

$$\begin{aligned} v(\eta) &= \xi \frac{\partial \eta}{\partial x} + \phi \frac{\partial \eta}{\partial u} = 0 \\ v(\zeta) &= \xi \frac{\partial \zeta}{\partial x} + \phi \frac{\partial \zeta}{\partial u} = 1. \end{aligned} \quad (3.13.7)$$

Here, the first equation means that η is an invariant of G , hence can be found by the method of characteristics indicated in (3.3.3). Once η is known, ζ can often be found by inspection. In any case, also the second equation can be solved by the method of characteristics. As indicated above, there is no guarantee that solving (3.13.7) is indeed easier than solving the original equation (3.13.1). For example, if

$$\frac{\phi(x, u)}{\xi(x, u)} = F(x, u) \quad (3.13.8)$$

then inserting $\phi = \xi \cdot F$ in (3.13.5) gives an identity, i.e., finding a symmetry in this case is exactly the same problem as solving the original equation.

3.13.1 Example. Consider the homogeneous equation

$$\frac{\partial u}{\partial x} = F\left(\frac{u}{x}\right).$$

This equation is invariant under the group of scaling transformations

$$G : (x, u) \mapsto (\lambda x, \lambda u), \quad \lambda > 0.$$

Indeed, by (3.13.4), the first prolongation of the generator $v = x\partial_x + u\partial_u$ is v itself and so one readily checks that (3.13.5) is satisfied. New coordinates satisfying (3.13.7) are given by

$$y = \frac{u}{x}, \quad w = \log x.$$

Then

$$\frac{\partial u}{\partial x} = \frac{du/dy}{dx/dy} = \frac{x(1+yw_y)}{xw_y} = \frac{1+yw_y}{w_y},$$

so the equation in the new coordinates becomes

$$\frac{dw}{dy} = \frac{1}{F(y) - y},$$

which can be solved by integration:

$$w = \int \frac{dy}{F(y) - y} + c.$$

3.13.2 Example. Consider the equation $P dx + Q du = 0$. We show that if it possesses a one-parameter symmetry with generator $v = \xi\partial_x + \phi\partial_u$, then the function

$$R(x, u) = \frac{1}{\xi(x, u)P(x, u) + \phi(x, u)Q(x, u)}$$

is an integrating factor.

To see this, note first that the given ODE is of the form (3.13.1) with $F = -P/Q$. Inserting this into (3.13.5) and recalling 3.12.4 and 3.12.2 it follows that v is a symmetry if and only if

$$\begin{aligned} & \left(\xi \frac{\partial P}{\partial x} + \phi \frac{\partial P}{\partial u} \right) Q - \left(\xi \frac{\partial Q}{\partial x} + \phi \frac{\partial Q}{\partial u} \right) P + \frac{\partial \phi}{\partial x} Q^2 \\ & - \left(\frac{\partial \phi}{\partial u} - \frac{\partial \xi}{\partial x} \right) PQ - \frac{\partial \xi}{\partial u} P^2 = 0. \end{aligned} \quad (3.13.9)$$

On the other hand, that R is an integrating factor means that

$$\frac{\partial}{\partial u}(RP) = \frac{\partial}{\partial x}(RQ),$$

and inserting R from above here gives

$$\begin{aligned} & R^2 \left(\phi \left(Q \frac{\partial P}{\partial u} - P \frac{\partial Q}{\partial u} \right) - \frac{\partial \xi}{\partial u} P^2 - \frac{\partial \phi}{\partial u} PQ \right) \\ & = R^2 \left(\xi \left(P \frac{\partial Q}{\partial x} - Q \frac{\partial P}{\partial x} \right) - \frac{\partial \xi}{\partial x} PQ - \frac{\partial \phi}{\partial x} Q^2 \right), \end{aligned}$$

which is equivalent to (3.13.9), as claimed.

Consider now a non-singular ODE of order n :

$$P(x, u^{(n)}) = P(x, u, u_x, \dots, u_n), \quad (3.13.10)$$

where $u_n = \frac{d^n u}{dx^n}$. We will show that knowledge of a one-parameter symmetry group G of (3.13.10) with generator v allows to reduce the order of the equation by one.

To see this, choose coordinates $y = \eta(x, u)$, $w = \zeta(x, u)$ as in (3.13.7) that straighten out v , so that $v = \partial_w$. Then by the chain rule, we can re-write the derivatives of u with respect to x in terms of y , w and the derivatives of w with respect to y :

$$\frac{d^k u}{dx^k} = \delta_k \left(y, w, \frac{dw}{dy}, \dots, \frac{d^k w}{dy^k} \right),$$

for suitable functions δ_k . Substituting these expressions back into (3.13.10), we obtain the equivalent n -th order equation

$$\tilde{P}(y, w^{(n)}) = \tilde{P}(y, w, w_y, \dots, w_n) = 0 \quad (3.13.11)$$

in the new coordinates. Now note that since (3.13.10) has G as a symmetry group, the same is true of the transformed system (3.13.11): this follows from the definition of a symmetry group and the fact that the prolongation of a group action according to (3.8.2) is equivariant under changing variables (again by the chain rule). In the new variables, $\text{pr}^{(n)}v = v = \partial_w$, and so the infinitesimal criterion (3.10.5) reduces to

$$\text{pr}^{(n)}v(\tilde{P}) = \frac{\partial \tilde{P}}{\partial w} = 0 \quad \text{whenever } \tilde{P}(y, w^{(n)}) = 0.$$

Since \tilde{P} is invariant under $G^{(n)}$, by 3.3.8 and 3.12.4 (and provided that G acts semi-regularly with orbits of constant dimension) there exists an equivalent equation that depends exclusively on a complete set of functionally independent invariants of $G^{(n)}$. In the present situation, such a set of invariants is obviously given by $y, w_y, w_{yy}, \dots, w_n$. Consequently, there is an equivalent equation

$$\hat{P}(y, w_y, \dots, w_n) = 0,$$

which is independent of w . Thus, setting $z = w_y$ we obtain an equivalent ODE of order $n - 1$, as claimed:

$$\hat{P}\left(y, z, \dots, \frac{d^{n-1}z}{dy^{n-1}}\right) = \hat{P}(y, z^{(n-1)}) = 0.$$

Given any solution $z = h(y)$ of this ODE, $w = \int h(y) dy + c$ is a solution to (3.13.11), and transforming back to (x, u) -coordinates gives a solution to (3.13.10).

3.13.3 Example. Any homogeneous linear ODE of second order

$$u_{xx} + p(x)u_x + q(x)u = 0 \quad (3.13.12)$$

is invariant under the scaling group $G : (x, u) \mapsto (x, \lambda u)$. In fact, the generator of G is $v = u\partial_u$, so $\text{pr}^{(2)}v = u\partial_u + u_x\partial_{u_x} + u_{xx}\partial_{u_{xx}}$, and so the infinitesimal criterion (3.10.5) is satisfied. Coordinates satisfying (3.13.7) are given by

$$y = x, \quad w = \log u,$$

so $v = \partial_w$. To re-express the equation in the new coordinates, note that

$$u = e^w, \quad u_x = w_x e^w, \quad u_{xx} = (w_{xx} + w_x^2)e^w,$$

giving

$$w_{xx} + w_x^2 + p(x)w_x + q(x) = 0,$$

which, as expected, is independent of w . Setting $z = w_x = u_x/u$ we therefore obtain the well-known transformation of (3.13.12) into the Riccati equation

$$z_x = -z^2 - p(x)z - q(x).$$

3.14 Differential invariants

In this section we look at a converse to the problem of determining the symmetry group of a differential equation, namely: Given a local transformation group, what is the most general type of differential equation that admits it as a symmetry group? We will give an answer to this question in the setting of ordinary differential equations.

Recall from 3.12.4 that a nondegenerate differential equation $P(x, u^{(n)}) = 0$ admits a local transformation group as a symmetry group if and only if the corresponding subvariety \mathcal{S}_P of $M^{(n)}$ is invariant under $\text{pr}^{(n)}G$. Provided that $\text{pr}^{(n)}G$ acts semi-regularly with orbits of constant dimension, 3.3.8 then implies that there is an equivalent equation $\tilde{P} = 0$ describing the subvariety \mathcal{S}_P and depending only on the functionally independent invariants of $\text{pr}^{(n)}G$. We therefore give a name to the invariants of $\text{pr}^{(n)}G$:

3.14.1 Definition. *Let G be a local transformation group acting on $M \subseteq X \times U$ and let $n \geq 1$. An n -th order differential invariant of G is a smooth function $\eta : M^{(n)} \rightarrow \mathbb{R}$ such that η is an invariant of $\text{pr}^{(n)}G$:*

$$\eta(\text{pr}^{(n)}g \cdot (x, u^{(n)})) = \eta(x, u^{(n)})$$

for all $(x, u^{(n)}) \in M^{(n)}$ and $g \in G$ such that $\text{pr}^{(n)}g \cdot (x, u^{(n)})$ is defined.

3.14.2 Example. For the rotation group $G = SO(2)$ on $X \times U = \mathbb{R}^2$ we have $v = -u\partial_x + x\partial_u$, and by (3.10.4) we have

$$\text{pr}^{(1)}v = -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x}$$

By definition, the first order differential invariants of G are the invariants of $\text{pr}^{(1)}G$, which can be calculated by the method of characteristics, cf. (3.3.3). This gives the following complete set of first order invariants of G :

$$y = \sqrt{x^2 + u^2}, \quad w = \frac{xu_x - u}{x + uu_x}.$$

Any other first order invariant must be a function of these (by 3.3.7).

Our next aim is to show that one can always calculate higher order differential invariants inductively from known ones of lower order. To show this we need an auxiliary result first.

3.14.3 Lemma. *Let $v = \xi\partial_x + \phi\partial_u$ be a vector field on $M \subseteq X \times U = \mathbb{R}^2$ and let $\zeta = \zeta(x, u^{(n)})$ be smooth. Then*

$$\text{pr}^{(n+1)}v(D_x\zeta) = D_x[\text{pr}^{(n)}v(\zeta)] - D_x\xi \cdot D_x\zeta. \quad (3.14.1)$$

Proof. By (3.10.29),

$$\text{pr}^{(n+1)}v(D_x\zeta) = \text{pr}^{(n+1)}v_Q(D_x\zeta) + \xi D_x^2\zeta,$$

and

$$D_x[\text{pr}^{(n)}v(\zeta)] = D_x[\text{pr}^{(n)}v_Q(\zeta)] + D_x(\xi D_x\zeta).$$

It therefore remains to show that

$$\text{pr}^{(n+1)}v_Q(D_x\zeta) = D_x[\text{pr}^{(n)}v_Q(\zeta)].$$

Using (3.10.30), we calculate:

$$\begin{aligned}
\text{pr}^{(n+1)}v_Q(D_x\zeta) &= \sum_{j=0}^{n+1} D_j Q \frac{\partial}{\partial u_j} \left(\frac{\partial \zeta}{\partial x} + \sum_{k=0}^n u_{k+1} \frac{\partial \zeta}{\partial u_k} \right) \\
&= \sum_{j=0}^{n+1} D_j Q \left(\frac{\partial^2 \zeta}{\partial u_j \partial x} + \sum_{k=0}^n \left(\delta_j^{k+1} \frac{\partial \zeta}{\partial u_k} + u_{k+1} \frac{\partial^2 \zeta}{\partial u_j \partial u_k} \right) \right) \\
&= \sum_{j=0}^n \left(D_{j+1} Q \frac{\partial \zeta}{\partial u_j} + D_j Q \left(\frac{\partial^2 \zeta}{\partial u_j \partial x} + \sum_{k=0}^n u_{k+1} \frac{\partial^2 \zeta}{\partial u_j \partial u_k} \right) \right) \\
&= D_x \left(\sum_{j=0}^n D_j Q \frac{\partial \zeta}{\partial u_j} \right) = D_x [\text{pr}^{(n)}v_Q(\zeta)],
\end{aligned}$$

where in the penultimate line we used that $\zeta = \zeta(x, u^{(n)})$, hence its derivative with respect to u_{n+1} vanishes. \square

3.14.4 Proposition. *Let G be a local one-parameter group of transformations acting on $M \subseteq X \times U = \mathbb{R}^2$. Let $y = \eta(x, u^{(n)})$ and $w = \zeta(x, u^{(n)})$ be n -th order differential invariants of G ($n \geq 1$). Then the derivative*

$$\frac{dw}{dy} = \frac{dw/dx}{dy/dx} \equiv \frac{D_x \zeta}{D_x \eta} \quad (3.14.2)$$

is a differential invariant of G of order $n+1$.

Proof. Note first that for any vector field Z and any smooth functions f, g we have the usual quotient rule $Z(f/g) = g^{-2}(Z(f)g - fZ(g))$, as follows from the derivation description of vector fields. Using this and (3.14.1), we obtain from (3.14.2):

$$\begin{aligned}
\text{pr}^{(n+1)}v \left(\frac{dw}{dy} \right) &= \frac{1}{(D_x \eta)^2} (\text{pr}^{(n+1)}v(D_x \zeta) \cdot D_x \eta - D_x \zeta \cdot \text{pr}^{(n+1)}v(D_x \eta)) \\
&= \frac{1}{(D_x \eta)^2} (D_x [\text{pr}^{(n)}v(\zeta)] \cdot D_x \eta - D_x \zeta \cdot D_x \xi \cdot D_x \zeta \cdot D_x \eta \\
&\quad - D_x \zeta \cdot D_x [\text{pr}^{(n)}v(\eta)] + D_x \zeta \cdot D_x \xi \cdot D_x \eta) = 0
\end{aligned}$$

since ζ and η are n -th order invariants, so $\text{pr}^{(n)}v(\zeta) = \text{pr}^{(n)}v(\eta) = 0$. \square

3.14.5 Corollary. *Let G be a local one-parameter group of transformations acting on $M \subseteq X \times U = \mathbb{R}^2$. Let $\{y = \eta(x, u), w = \zeta(x, u, u_x)\}$ be a complete set of functionally independent invariants of $\text{pr}^{(1)}G$. Then (locally around any point) the derivatives*

$$y, w, \frac{dw}{dy}, \dots, \frac{d^{n-1}w}{dy^{n-1}}$$

form a complete set of functionally independent invariants for $\text{pr}^{(n)}G$ for $n \geq 1$.

Proof. Successively applying 3.14.4 it follows that the derivatives up to order k are invariants of $\text{pr}^{(k)}G$, hence also of any higher prolongation of G (by (3.8.4)). So it only remains to show that they are functionally independent. But this is immediate because $\frac{d^k w}{dy^k}$ depends explicitly on u_{k+1} , hence is independent of the previously

constructed invariants $y, w, \frac{\partial w}{\partial y}, \dots, \frac{d^{k-1}w}{dy^{k-1}}$, since those only depend on x, u, \dots, u_k . By 3.3.7, since G has one-dimensional orbits and $\dim X \times U^{(n)} = n+2$, completeness follows. \square

3.14.6 Example. Applying the previous result to the first order invariants of the rotation group given in 3.14.2 it follows that y, w , and

$$\frac{dw}{dy} = \frac{dw/dx}{dy/dx} = \frac{\sqrt{x^2 + u^2}}{(x + uu_x)^3} [(x^2 + u^2)u_{xx} - (1 + u_x^2)(xu_x - u)]$$

together form a complete set of functionally independent invariants for $\text{pr}^{(2)}G$. Thus any other second order differential invariant can be written as a smooth function of these invariants. For example, the curvature κ from (3.10.25) is an invariant of $\text{pr}^{(2)}G$ (as one checks by verifying that $\text{pr}^{(2)}v(\kappa) = 0$), and in fact

$$\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} = \frac{w_y}{(1 + w^2)^{3/2}} + \frac{w}{y(1 + w^2)^{1/2}}.$$

The important point to note now is that once the differential invariants of a local group of transformations G on $M \subseteq X \times U = \mathbb{R}^2$ are known, we can determine all ODEs that admit G as a symmetry group, i.e., all ODEs that can be integrated using G :

3.14.7 Theorem. *Let G be a connected local transformation group acting on $M \subseteq X \times U = \mathbb{R}^2$ and let $\eta^1(x, u^{(n)}), \dots, \eta^k(x, u^{(n)})$ be a complete set of functionally independent n -th order differential invariants of G .*

- (i) *If $\text{pr}^{(n)}G$ acts semi-regularly on $M^{(n)}$, then an n -th order nondegenerate differential equation $P(x, u^{(n)}) = 0$ admits G as a symmetry group if and only if, locally around any point in $\mathcal{S}_P = \{(x, u^{(n)}) \mid P(x, u^{(n)}) = 0\} \subseteq M^{(n)}$, there is an equivalent equation*

$$\tilde{P}(\eta^1(x, u^{(n)}), \dots, \eta^k(x, u^{(n)})) = 0$$

involving only the differential invariants of G .

- (ii) *If G is a one-parameter group of transformations, any nondegenerate n -th order differential equation having G as a symmetry group is equivalent to an $(n-1)$ -st order equation*

$$\tilde{P}\left(y, w, \frac{dw}{dy}, \dots, \frac{d^{n-1}w}{dy^{n-1}}\right) = 0,$$

where $y = \eta(x, u)$, $w = \zeta(x, u, u_x)$ form a complete set of functionally independent invariants of $\text{pr}^{(1)}G$.

Proof. (i) Clearly if there is an equivalent equation depending only on invariants then \mathcal{S}_P itself is locally invariant. Since P is non-degenerate, \mathcal{S} is a closed regular submanifold of $M^{(n)}$, so the proof of 3.2.13 shows that this implies that \mathcal{S} is invariant under $\text{pr}^{(n)}G$, which in turn yields that G is a symmetry group of P . Conversely, if G is a symmetry group then the proof of 3.12.4 shows that $\text{pr}^{(n)}G$ leaves \mathcal{S}_P invariant. The claim then follows from 3.3.8.

- (ii) This is immediate from (i) and 3.14.5. \square

3.14.8 Example. Continuing our analysis of the rotation group from 3.14.6, we can use 3.14.7 to find the most general first and second order ODEs that admit the rotation group as a symmetry group. By 3.14.7 (ii), any such first order equation is equivalent to one involving only the first order invariants given in 3.14.2. If we solve for w it follows that any such equation is of the form

$$\frac{xu_x - u}{x + uu_x} = F(\sqrt{x^2 + u^2}).$$

Analogously, any second order equation invariant under rotations is equivalent to one only involving y , w , and, say, the curvature $\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$, hence is of the general form

$$u_{xx} = (1 + u_x^2)^{3/2} F\left(\sqrt{x^2 + u^2}, \frac{xu_x - u}{x + uu_x}\right).$$

We may also employ 3.14.7 to obtain an alternative way of reducing the order of a given differential equation $P(x, u^{(n)}) = 0$ once we know a one-parameter symmetry group. In fact, we know from 3.14.7 (ii) that the equation must be equivalent to one involving only $y, w, \frac{dw}{dy}, \dots, \frac{d^{n-1}w}{dy^{n-1}}$. But this latter equation automatically is of order $n - 1$. Thus simply by re-expressing the equation in terms of the differential invariants already reduces the order by 1. Furthermore, once the solution $w = h(y)$ of the reduced equation is known, the solution of the original equation is found by solving the auxiliary first order equation

$$\zeta(x, u, u_x) = h(\eta(x, u)) \quad (3.14.3)$$

obtained by substituting for y and w their expression in terms of x and u . As (3.14.3) depends only on the invariants η, ζ of $\text{pr}^{(1)}G$, it has G as a one-parameter symmetry group, hence can be integrated using the methods described above for first order equations.

3.14.9 Example. The second order equation

$$x^2 u_{xx} + xu_x^2 = uu_x \quad (3.14.4)$$

is invariant under the scaling group $G : (x, u) \mapsto (\lambda x, \lambda u)$. In fact, the infinitesimal generator of G is $v = \frac{d}{d\lambda} \Big|_{\lambda=1} (\lambda x, \lambda u) = x\partial_x + u\partial_u$. Hence (3.10.19) gives

$$\text{pr}^{(2)}v = x\partial_x + u\partial_u - u_{xx}\partial_{u_{xx}},$$

so

$$\text{pr}^{(2)}v(x^2 u_{xx} + xu_x^2 - uu_x) = 2x^2 u_{xx} + xu_x^2 - uu_x - x^2 u_{xx} = 0$$

for any $(x, u, u_x, u_{xx}) \in \mathcal{S}_P$. By 3.14.5 and 3.13.1, a complete set of functionally independent invariants of second order is given by

$$y = \frac{u}{x}, \quad w = u_x, \quad \frac{dw}{dy} = \frac{dw/dx}{dy/dx} = \frac{x^2 u_{xx}}{xu_x - u}.$$

Inserting this into (3.14.4) gives the transformed equation

$$(w - y) \frac{dw}{dy} + w^2 = yw,$$

which, as expected, is of first order. This has two families of solutions: either $w = y$ (which is a singular solution), or $\frac{dw}{dy} = -w$, which implies $w = ce^{-y}$. Re-substituting gives

$$\frac{du}{dx} = \frac{u}{x}, \quad \text{or} \quad \frac{du}{dx} = ce^{-u/x}.$$

This results in the one-parameter family of singular solutions $u = kx$, and the implicit solutions

$$\int \frac{dy}{ce^{-y} - y} = \log x + k,$$

with $y = u/x$, which constitutes the general solution to the original equation (3.14.4).

3.14.10 Example. We return to the two-dimensional heat equation $u_t = u_{xx}$, whose symmetry group we calculated in 3.11.1, and use the methods of this section to determine the general form of *traveling wave solutions* to the heat equation. In general, a traveling wave solution to a differential equation is one which is invariant under a translation group. Here we consider the symmetry of the heat equation given by the translation group

$$(x, t, u) \mapsto (x + c\varepsilon, t + \varepsilon, u), \quad \varepsilon \in \mathbb{R},$$

with generator $v = \partial_t + c\partial_x$, for c some fixed constant. Global invariants are given by

$$y = x - ct, \quad v = u. \quad (3.14.5)$$

Thus any group-invariant solution is of the form $v = h(y)$, or $u = h(x - ct)$, which is a function that doesn't change its shape along the straight line $x - ct = \text{const}$. If we express the derivatives of u in terms of the new variables we obtain

$$u_t = -cv_y, \quad u_x = v_y, \quad u_{xx} = v_{yy}.$$

Inserting into the heat equation, we arrive at the ODE determining all traveling wave solutions of the above form:

$$-cv_y = v_{yy},$$

with general solution

$$v(y) = ke^{-cy} + l$$

(k, l arbitrary constants). If we now substitute back into the heat equation we obtain the general form of traveling wave solutions:

$$u(x, t) = ke^{-c(x-ct)} + l.$$

Much more can be said about group-invariant solutions of PDEs. In fact, there is an entire theory that allows to calculate solutions invariant under any subgroup of the full symmetry group of any given system of differential equations. While these methods also basically rest on choosing new variables from invariants of the given sub-group, the case of higher (than one) dimensional symmetry groups is significantly more involved than what we considered in this section. For a precise formulation, the theory of extended jet bundles and group invariant prolongation is required. We refer to [14, Ch. 3] for an in-depth study.

Chapter 4

Variational symmetries

Conservation laws (like conservation of energy, momentum, etc.) play a central role in physics. In contemporary theoretical physics one usually formulates the governing principles of a theory in the form of a variational principle – roughly, this says that some relevant quantity is to take an optimal value. It is a deep and far-reaching discovery of Emmy Noether (1918) that for such systems, every conservation law comes from a corresponding symmetry property. In the present chapter we give an introduction to this field.

4.1 Calculus of variations

Let $X = \mathbb{R}^p$, with coordinates (x^1, \dots, x^p) , and $U = \mathbb{R}^q$ with coordinates (u^1, \dots, u^q) . Also, let Ω be a connected open subset of X with smooth boundary $\partial\Omega$. By a *variational problem* we mean the problem of finding extrema (maxima, minima, or stationary points) of the functional

$$\mathcal{L}[u] = \int_{\Omega} L(x, u^{(n)}(x)) dx$$

in some class of admissible functions $u = f(x)$ on Ω . The integrand $L : X \times U^{(n)} \rightarrow \mathbb{R}$ is called the *Lagrangian* of the variational problem \mathcal{L} . The class of admissible functions varies with the problem under consideration (e.g., through the imposition of various boundary or regularity conditions). We will confine ourselves here to studying smooth variational problems. The calculus of variations is a vast subject and we will here barely be able to scratch the surface, with a focus on symmetry methods. For a gentle introduction see, e.g., [16], a rather comprehensive treatise is [5, 6].

4.1.1 Example. (i) To find the shortest curve $u = f(x)$ connecting two points (a, b) , (c, d) in the plane, one needs to minimize the length of $u = f(x)$:

$$\mathcal{L}[u] = \int_a^c \sqrt{1 + u_x^2} dx.$$

(ii) More generally, if (M, g) is a smooth Riemannian manifold and $x, y \in M$ then the problem of finding the shortest curve from x to y is a variational problem, namely that of minimizing the arc-length of curves $u : [0, 1] \rightarrow M$ connecting x and y :

$$\mathcal{L}[u] = \int_0^1 \sqrt{g(u'(t), u'(t))} dt.$$

Solutions to this problem are called geodesics of (M, g) .

The basic approach to finding extremals of variational problems is very similar to that of extremizing smooth functions in basic real analysis. There, given a function $f: \Omega \rightarrow \mathbb{R}$, one notes that if x is an extremum of f then for any y , the one-dimensional function $\varepsilon \mapsto f(x + \varepsilon y)$ must have an extremal. Therefore, we must have

$$0 = \left. \frac{d}{d\varepsilon} \right|_0 f(x + \varepsilon y) = \langle \nabla f(x), y \rangle \quad (y \in \mathbb{R}^p),$$

and so the gradient ∇f of f at x vanishing is a necessary condition for x being an extremum of f .

For functionals $\mathcal{L}[u]$, the role of the gradient is taken over by what is called the *variational derivative*. Moreover, the inner product on \mathbb{R}^n gets replaced by the L^2 -inner product of functions $f, g: \Omega \rightarrow \mathbb{R}^q$:

$$\langle f, g \rangle = \int_{\Omega} f(x) \cdot g(x) dx = \int_{\Omega} \sum_{\alpha=1}^q f^{\alpha}(x) g^{\alpha}(x) dx.$$

With these notations we can define:

4.1.2 Definition. Let $\mathcal{L}[u]$ be a variational problem. The variational derivative of \mathcal{L} is the unique q -tuple

$$\delta \mathcal{L}[u] = (\delta_1 \mathcal{L}[u], \dots, \delta_q \mathcal{L}[u]),$$

$\delta \mathcal{L}: \mathcal{C}^{\infty}(\Omega) \rightarrow \mathcal{C}^{\infty}(\Omega)^q$ with the property that

$$\left. \frac{d}{d\varepsilon} \right|_0 \mathcal{L}[f + \varepsilon \eta] = \int_{\Omega} \delta \mathcal{L}[f](x) \cdot \eta(x) dx \quad (4.1.1)$$

for every smooth function f on Ω and any compactly supported smooth function $\eta \in \mathcal{D}(\Omega)^q$ such that $f + \varepsilon \eta$ is itself admissible. $\delta_{\alpha} \mathcal{L}$ is called the variational derivative of \mathcal{L} with respect to u^{α} .

Here, by $\mathcal{D}(\Omega)$ we denote the space of test functions on Ω , $\mathcal{D}(\Omega) = \{f \in \mathcal{C}^{\infty}(\Omega) \mid \text{supp}(f) \Subset \Omega\}$.

4.1.3 Proposition. If $u = f(x)$ is an extremal of $\mathcal{L}[u]$, then

$$\delta \mathcal{L}[f](x) = 0 \quad \forall x \in \Omega. \quad (4.1.2)$$

Proof. For any $\eta \in \mathcal{D}(\Omega)^q$ and for all ε small, $f + \varepsilon \eta$ is admissible, and by assumption, $\varepsilon \mapsto \mathcal{L}[f + \varepsilon \eta]$ must have an extremum at $\varepsilon = 0$. Hence by classical real analysis,

$$0 = \left. \frac{d}{d\varepsilon} \right|_0 \mathcal{L}[f + \varepsilon \eta] = \int_{\Omega} \delta \mathcal{L}[f](x) \cdot \eta(x) dx.$$

As this has to hold for any η , the claim follows. \square

Using the notion of total derivative, cf. (3.10.15), we now wish to derive an explicit formula for the variational derivative. First,

$$\begin{aligned} \left. \frac{d}{d\varepsilon} \right|_0 \mathcal{L}[f + \varepsilon \eta] &= \int_{\Omega} \left. \frac{d}{d\varepsilon} \right|_0 L(x, \text{pr}^{(n)}(f + \varepsilon \eta)(x)) dx \\ &= \int_{\Omega} \left(\sum_{\alpha, j} \frac{\partial L}{\partial u_j^{\alpha}}(x, \text{pr}^{(n)} f(x)) \cdot \partial_j \eta^{\alpha}(x) \right) dx. \end{aligned}$$

Since η has compact support, we may apply the divergence theorem to this expression, with all boundary terms vanishing. Note also that when a partial derivative ∂_{x_j} is applied to $\partial L / \partial u_j^\alpha$ then the result can be expressed by a total derivative, so

$$\left. \frac{d}{d\varepsilon} \right|_0 \mathcal{L}[f + \varepsilon\eta] = \int_{\Omega} \sum_{\alpha=1}^q \left[\sum_J (-D)_J \frac{\partial L}{\partial u_j^\alpha}(x, \text{pr}^{(n+\sharp J)} f(x)) \right] \eta^\alpha(x) dx. \quad (4.1.3)$$

Here, for $J = (j_1, \dots, j_k)$ we set $(-D)_J := (-D_{j_1})(-D_{j_2}) \dots (-D_{j_k})$. This formula is the first instance of the following operator that will play a central role below.

4.1.4 Definition. For $1 \leq \alpha \leq q$, the α -th Euler operator is defined by

$$E_\alpha = \sum_J (-D)_J \frac{\partial}{\partial u_j^\alpha}, \quad (4.1.4)$$

where the sum extends over all $J = (j_1, \dots, j_k)$ with $1 \leq j_k \leq p$, $k \geq 0$. Moreover, we set $E(L) := (E_1(L), \dots, E_q(L))$.

Although the above sum is formally infinite, whenever E_α is applied to some concrete $L(x, u^{(n)})$ only finitely many terms are required.

In terms of the Euler operator, (4.1.3) says that

$$\delta \mathcal{L} = E(L)$$

Together with 4.1.2, this gives the classical necessary condition for extremals of a variational problem:

4.1.5 Theorem. If $u = f(x)$ is a smooth extremal of the variational problem $\mathcal{L}[u] = \int_{\Omega} L(x, u^{(n)}) dx$ then it must be a solution of the Euler–Lagrange equations

$$E_\alpha(L) = 0, \quad \alpha = 1, \dots, q.$$

□

4.1.6 Example. Let $p = q = 1$, i.e., we consider a variational problem for a real-valued function of one variable. In this case,

$$E = \sum_{j=0}^{\infty} (-D_x)^j \frac{\partial}{\partial u_j} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - \dots$$

Thus the Euler–Lagrange equation for an n -th order variational problem

$$\mathcal{L}[u] = \int_a^b L(x, u^{(n)}) dx$$

is given by (setting $u_j = \frac{\partial^j u}{\partial x^j}$)

$$0 = E(L) = \frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \dots + (-1)^n D_x^n \frac{\partial L}{\partial u_n}.$$

This is an ODE of order (at most) $2n$. The most common case of a first order variational problem $L = L(x, u, u_x)$ gives

$$0 = E(L) = \frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} = \frac{\partial L}{\partial u} - \frac{\partial^2 L}{\partial x \partial u_x} - u_x \frac{\partial^2 L}{\partial u \partial u_x} - u_{xx} \frac{\partial^2 L}{\partial u_x^2}.$$

In particular, for the curve length problem 4.1.1 the Euler–Lagrange equation reads

$$-D_x \left(\frac{u_x}{\sqrt{1+u_x^2}} \right) = -\frac{u_{xx}}{(1+u_x^2)^{3/2}} = 0,$$

i.e., $u_{xx} = 0$. As geometrically expected, the solutions are straight lines $u = kx + d$.

4.1.7 Example. The Dirichlet principle: Consider the variational problem of minimizing the total energy (kinetic plus potential) of some system in the form

$$\mathcal{L}[u] = \int_{\Omega} \frac{1}{2} |\nabla u|^2 - uh \, dx$$

with some external potential h . Then the Euler–Lagrange equation reads

$$0 = E(L) = \frac{\partial L}{\partial u} - \sum_{i=1}^p D_i \frac{\partial L}{\partial u_i} = \frac{\partial L}{\partial u} - \sum_{i=1}^p D_i u_i = -h - \Delta u,$$

i.e., we obtain the Poisson equation $-\Delta u = h$.

4.2 Variational symmetries

In this section we want to develop a notion of symmetry that applies to variational problems, similar to the symmetry groups of differential equations studied in the previous chapter. Consider a variational problem

$$\mathcal{L}[u] = \int_{\Omega_0} L(x, u^{(n)}) \, dx. \quad (4.2.1)$$

Let G be a local transformation group on an open subset M of $\Omega_0 \times U \subseteq X \times U$. If $u = f(x)$ is a smooth function on a sufficiently small subdomain $\Omega \subseteq \Omega_0$ such that the graph of f lies in M then any transformation $g \in G$ sufficiently close to the identity will transform f into another smooth function $\tilde{u} = \tilde{f}(\tilde{x}) = (g \cdot f)(\tilde{x})$ defined on some $\tilde{\Omega} \subseteq \Omega_0$. Therefore, we define:

4.2.1 Definition. A local transformation group G acting on $M \subseteq \Omega_0 \times U$ is called a variational symmetry of the functional (4.2.1) if whenever Ω is a subdomain with $\overline{\Omega} \subseteq \Omega_0$, $u = f(x)$ is a function defined over Ω with graph contained in M , and $g \in G$ is such that $\tilde{u} = \tilde{f}(\tilde{x}) = (g \cdot f)(\tilde{x})$ is a well-defined function defined on $\tilde{\Omega} \subseteq \Omega_0$, then

$$\int_{\tilde{\Omega}} L(\tilde{x}, \text{pr}^{(n)} \tilde{f}(\tilde{x})) \, d\tilde{x} = \int_{\Omega} L(x, \text{pr}^{(n)} f(x)) \, dx. \quad (4.2.2)$$

Our first aim, in line with our previous considerations on symmetries of differential equations, is to derive an infinitesimal criterion for variational symmetries. For this, we need a few preparations.

To begin with, we need to study how a variational problem transforms under the action of a local transformation group. Let

$$\tilde{x} = \Xi(x, u), \quad \tilde{u} = \Phi(x, u) \quad (4.2.3)$$

be any change of variables. Then there is an induced change of variables

$$\tilde{u}^{(n)} = \Phi^{(n)}(x, u^{(n)})$$

for the derivatives, given by prolongation (i.e., by differentiating (4.2.3) by the chain rule). If the conditions of the inverse function theorem are satisfied then (4.2.3) determines a transformed function $\tilde{u} = \tilde{f}(\tilde{x})$. Thereby, any functional

$$\mathcal{L}[f] = \int_{\Omega} L(x, \text{pr}^{(n)} f(x)) \, dx$$

is transformed into a new functional

$$\tilde{\mathcal{L}}[\tilde{f}] = \int_{\tilde{\Omega}} \tilde{L}(\tilde{x}, \text{pr}^{(n)} \tilde{f}(\tilde{x})) d\tilde{x}.$$

Here, the new domain $\tilde{\Omega} = \{\tilde{x} = \Xi(x, f(x)) \mid x \in \Omega\}$ depends both on Ω and on f . By the change of variables formula we get

$$L(x, \text{pr}^{(n)} f(x)) = \tilde{L}(\tilde{x}, \text{pr}^{(n)} \tilde{f}(\tilde{x})) \det J(x, \text{pr}^{(1)} f(x)), \quad (4.2.4)$$

where J is the Jacobian matrix with entries

$$J^{ij}(x, \text{pr}^{(1)} f(x)) = \frac{\partial}{\partial x^j} \Xi^i(x, f(x)) = D_j \Xi^i(x, \text{pr}^{(1)} f(x)) \quad i, j = 1, \dots, p, \quad (4.2.5)$$

where, for simplicity, we assume that $\det J(x) > 0$, otherwise we have to add absolute value signs.

4.2.2 Definition. Let $x = (x^1, \dots, x^p)$ and let $P = P(x, u^{(n)}) = (P_1(x, u^{(n)}), \dots, P_p(x, u^{(n)}))$ be a p -tuple of smooth functions. Then the total divergence of P is

$$\text{Div } P = D_1 P_1 + \dots + D_p P_p,$$

where D_j denotes the total derivative with respect to x^j .

With these notations, we have:

4.2.3 Lemma. Let $v = \sum_i \xi^i \partial_{x^i} + \sum_\alpha \phi^\alpha \partial_{u^\alpha}$ be a vector field on $M \subseteq \Omega_0 \times U$ and set $g_\varepsilon := \text{Fl}_\varepsilon^v = (\Xi_{g_\varepsilon}, \Phi_{g_\varepsilon})$. Denoting by J_{g_ε} the Jacobian (4.2.5) corresponding to Ξ_{g_ε} , we have

$$\frac{d}{d\varepsilon} (\det J_{g_\varepsilon}(x, u^{(1)})) = (\text{Div } \xi)(\text{pr}^{(1)} g_\varepsilon \cdot (x, u^{(1)})) \cdot \det J_{g_\varepsilon}(x, u^{(1)}). \quad (4.2.6)$$

Proof. Recall first that for matrices A, B with A invertible, the derivative of the determinant function at A in the direction B is given by $\det'(A)(B) = \det(A) \cdot \text{tr}(A^{-1}B)$. Consequently, for any fixed function u we get

$$\frac{d}{d\varepsilon} \det J_{g_\varepsilon} = \det'(J_{g_\varepsilon}) \left(\frac{d}{d\varepsilon} J_{g_\varepsilon} \right) = \det J_{g_\varepsilon} \cdot \text{tr} \left(J_{g_\varepsilon}^{-1} \frac{d}{d\varepsilon} J_{g_\varepsilon} \right). \quad (4.2.7)$$

By (4.2.5), $J_{g_\varepsilon}(x, u^{(1)}(x)) = \partial_x(\Xi_{g_\varepsilon}(x, u(x)))$ (with ∂_x abbreviating the x -gradient, and similar for ∂_1, ∂_2 below), so

$$\begin{aligned} \frac{d}{d\varepsilon} (J_{g_\varepsilon}(x, u^{(1)})(x)) &= \partial_x \left(\frac{d}{d\varepsilon} \Xi_{g_\varepsilon}(x, u(x)) \right) = \partial_x (\xi(\Xi_{g_\varepsilon}(x, u(x)), \Phi_{g_\varepsilon}(x, u(x)))) \\ &= (\partial_1 \xi)(\Xi_{g_\varepsilon}(x, u(x)), \Phi_{g_\varepsilon}(x, u(x))) \partial_x(\Xi_{g_\varepsilon}(x, u(x))) \\ &\quad + (\partial_2 \xi)(\Xi_{g_\varepsilon}(x, u(x)), \Phi_{g_\varepsilon}(x, u(x))) \partial_x(\Phi_{g_\varepsilon}(x, u(x))) \\ &= [(\partial_1 \xi)(\dots) + \partial_2 \xi(\dots) \cdot \partial_x(\Phi_{g_\varepsilon}(x, u(x))) \cdot \partial_x(\Xi_{g_\varepsilon}(x, u(x)))^{-1}] \partial_x(\Xi_{g_\varepsilon}(x, u(x))) \end{aligned}$$

Now let \tilde{u} be the transformed function $g_\varepsilon \cdot u$, so that with $\tilde{x}(x) = \Xi_{g_\varepsilon}(x, u(x))$ we have $\tilde{u}(\tilde{x}(x)) = \Phi_{g_\varepsilon}(x, u(x))$. Then

$$\partial_x(\Phi_{g_\varepsilon}(x, u(x))) = \partial_x(\tilde{u} \circ \tilde{x})(x) = \partial_{\tilde{x}} \tilde{u}(\tilde{x}(x)) \cdot \partial_x \tilde{x}(x) = \partial_{\tilde{x}} \tilde{u}(\tilde{x}(x)) \cdot \partial_x(\Xi_{g_\varepsilon}(x, u(x))),$$

so, together with the above calculations we obtain

$$\begin{aligned} \frac{d}{d\varepsilon} (J_{g_\varepsilon}(x, u^{(1)})(x)) &= [(\partial_1 \xi)(\tilde{x}, \tilde{u}(\tilde{x})) + (\partial_2 \xi)(\tilde{x}, \tilde{u}(\tilde{x})) \cdot \partial_{\tilde{x}} \tilde{u}(\tilde{x}(x))] \cdot J_{g_\varepsilon}(x, u^{(1)}(x)) \\ &= D_{\tilde{x}} \xi(\text{pr}^{(1)} g_\varepsilon(x, u^{(1)}(x))) \cdot J_{g_\varepsilon}(x, u^{(1)}(x)). \end{aligned}$$

Finally, using (4.2.7), and the fact that $\text{tr}(AB) = \text{tr}(BA)$, we arrive at

$$\begin{aligned} \frac{d}{d\varepsilon}(\det J_{g_\varepsilon}(x, u^{(1)}(x))) &= \det(J_{g_\varepsilon}(x, u^{(1)}(x))) \cdot \text{tr}(D_{\tilde{x}}\xi(\text{pr}^{(1)}g_\varepsilon(x, u^{(1)}(x)))) \\ &= \det(J_{g_\varepsilon}(x, u^{(1)}(x))) \cdot (\text{Div } \xi)(\text{pr}^{(1)}g_\varepsilon(x, u^{(1)}(x))). \end{aligned}$$

□

The infinitesimal criterion for variational symmetries now is given by the following result.

4.2.4 Theorem. *Let G be a connected local transformation group acting on $M \subseteq \Omega_0 \times U$. Then G is a variational symmetry group of the variational problem (4.2.1) if and only if*

$$\text{pr}^{(n)}v(L) + L \cdot \text{Div } \xi = 0 \quad (4.2.8)$$

for all $(x, u^{(n)}) \in M^{(n)}$ and every infinitesimal generator

$$v = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

of G .

Proof. As usual, for $g \in G$ we write $(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\Xi_g(x, u), \Phi_g(x, u))$. By definition of a variational symmetry, see (4.2.2), we get

$$\begin{aligned} \int_{\Omega} L(x, \text{pr}^{(n)}f(x)) dx &= \int_{\tilde{\Omega}} L(\tilde{x}, \text{pr}^{(n)}\tilde{f}(\tilde{x})) d\tilde{x} \\ &\stackrel{(4.2.4)}{=} \int_{\Omega} L(\tilde{x}(x), \text{pr}^{(n)}\tilde{f}(\tilde{x}(x))) \det J_g(x, \text{pr}^{(1)}f(x)) dx \end{aligned}$$

Since this equality has to hold for all subdomains Ω and all functions f , the integrands must agree pointwise, so g is a variational symmetry if and only if

$$L(\text{pr}^{(n)}g \cdot (x, u^{(n)})) \det J_g(x, u^{(1)}) = L(x, u^{(n)}) \quad \forall (x, u^{(n)}) \in M^{(n)} \quad (4.2.9)$$

for all g such that both sides are defined. We now insert $g = g_\varepsilon = \text{Fl}_\varepsilon^v$ and differentiate with respect to ε , which by (4.2.6) and the product rule yields

$$(\text{pr}^{(n)}v(L) + L \cdot \text{Div } \xi)|_{(\text{pr}^{(n)}g_\varepsilon \cdot (x, u^{(n)}))} \cdot \det J_{g_\varepsilon}(x, u^{(1)}) = 0 \quad (4.2.10)$$

Setting $\varepsilon = 0$, i.e., $g_\varepsilon = \text{id}$, this formula reduces to (4.2.8), proving necessity.

Conversely, if (4.2.8) is satisfied then also (4.2.10) holds for ε sufficiently small. Therefore, the derivative of (4.2.9) (with $g = g_\varepsilon$) with respect to ε vanishes identically. Integrating from 0 to ε then shows that (4.2.9) holds for any g_ε . Since G is connected the result then follows since any $g \in G$ can be written as a product of such g_ε . □

4.2.5 Example. Returning to (4.1.1) (i), where $\mathcal{L}[u] = \int_a^c \sqrt{1 + u_x^2} dx$, we expect that rotations should be variational symmetries, as they leave curve lengths invariant. To verify this, let $v = -u\partial_x + x\partial_u$ be a generator of the rotation group on \mathbb{R}^2 . Then by (3.10.4),

$$\text{pr}^{(1)}v = -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x}.$$

In particular $\xi = -u$, and we calculate

$$\text{pr}^{(1)}v(L) + L \cdot D_x \xi = (1 + u_x^2) \frac{\partial}{\partial u_x} \sqrt{1 + u_x^2} - \sqrt{1 + u_x^2} \cdot u_x = 0,$$

so indeed (4.2.8) is satisfied.

Next, we want to clarify the relationship between variational symmetries of some variational problem and symmetries of the corresponding Euler–Lagrange equations. To this end, we first need to understand how the Euler–Lagrange equations of variational problems transform under a change of variables.

4.2.6 Proposition. *Let $L(x, u^{(n)})$, $\tilde{L}(\tilde{x}, \tilde{u}^{(n)})$ be two Lagrangians related by a change of variables given by (4.2.3), (4.2.4), i.e.,*

$$\tilde{x} = \Xi(x, u), \quad \tilde{u} = \Phi(x, u)$$

and

$$L(x, u^{(n)}) = \tilde{L}(\tilde{x}, \tilde{u}^{(n)}) \det J(x, u^{(1)}).$$

Then

$$E_{u^\alpha}(L)(x, u^{(2n)}) = \sum_{\beta=1}^q F_{\alpha\beta}(x, u^{(1)}) E_{\tilde{u}^\beta}(\tilde{L})(\tilde{x}, \tilde{u}^{(2n)}), \quad \alpha = 1, \dots, q, \quad (4.2.11)$$

where $F_{\alpha\beta}$ is the determinant of the following $(p+1) \times (p+1)$ -matrix:

$$F_{\alpha\beta} = \det \begin{pmatrix} D_1 \Xi^1 & \dots & D_p \Xi^1 & \frac{\partial \Xi^1}{\partial u^\alpha} \\ \vdots & & \vdots & \vdots \\ D_1 \Xi^p & \dots & D_p \Xi^p & \frac{\partial \Xi^p}{\partial u^\alpha} \\ D_1 \Phi^\beta & \dots & D_p \Phi^\beta & \frac{\partial \Phi^\beta}{\partial u^\alpha} \end{pmatrix}. \quad (4.2.12)$$

Proof. Let $(x_0, u_0^{(2n)})$ be any fixed point in $X \times U^{(2n)}$ and let f be a smooth function defined near x_0 with $(x_0, u_0^{(2n)}) = (x_0, \text{pr}^{(2n)} f(x_0))$. By the inverse function theorem, on an open ball Ω around x_0 , f is transformed by our change of variables into a function $\tilde{u} = \tilde{f}(\tilde{x})$. If $\eta \in \mathcal{D}(\Omega)^q$ and ε is sufficiently small, then the perturbations $u_\varepsilon = f(x, \varepsilon) = f(x) + \varepsilon \eta(x)$, required for calculating variational derivatives, are transformed into functions $\tilde{u} = \tilde{f}(\tilde{x}, \varepsilon)$, which are determined implicitly by

$$\tilde{x} = \Xi(x, f(x) + \varepsilon \eta(x)) =: \Xi_\varepsilon(x), \quad \tilde{u} = \Phi(x, f(x) + \varepsilon \eta(x)) =: \Phi_\varepsilon(x). \quad (4.2.13)$$

Again the inverse function theorem determines the domain of each $\tilde{f}(\cdot, \varepsilon)$. More precisely, by [1, Suppl. 2.5 A], the minimal radius of a ball around $\tilde{x}_\varepsilon := \Xi(x, f(x_0) + \varepsilon \eta(x_0))$ where $\tilde{f}(\cdot, \varepsilon)$ is defined can be estimated from below in terms of the second derivatives of the maps Ξ_ε from (4.2.13) in a neighborhood of x_0 as well as the norm of the inverse of their Jacobians at x_0 . Since η is compactly supported, $\bar{\Omega}$ is compact, and $\tilde{x}_\varepsilon \rightarrow \tilde{x}_0$ as $\varepsilon \rightarrow 0$, by choosing ε small we can therefore achieve that all these balls contain a fixed neighborhood of \tilde{x}_0 , which we then take as a common domain $\tilde{\Omega}$ of all $\tilde{f}(\cdot, \varepsilon)$, independently of ε . In fact, since the radius of the ball Ω around x_0 is itself determined by the mentioned estimates (with $\varepsilon = 0$), we may in addition achieve that $\text{supp}(\eta)$ is contained in $\Xi_\varepsilon^{-1}(\tilde{\Omega})$ for ε small.

Now set $\tilde{\eta}(\tilde{x}, \varepsilon) := \tilde{f}(\tilde{x}, \varepsilon) - \tilde{f}(\tilde{x})$ and note that $\Xi_\varepsilon(x) = \Xi_0(x)$ as well as $\Phi_\varepsilon(x) = \Phi_0(x)$, and thereby $\tilde{f}(\Xi_\varepsilon(x), \varepsilon) = \tilde{f}(\Xi_0(x))$ for $x \notin \text{supp}(\eta)$. This implies that $\tilde{\eta}(\tilde{y}, \varepsilon) = 0$ for $\tilde{y} \notin \Xi_\varepsilon(\text{supp}(\eta))$. Again due to the compactness of $\text{supp}(\eta)$ this shows that, for ε sufficiently small, all $\tilde{\eta}(\cdot, \varepsilon)$ have compact support contained in $\tilde{\Omega}$. The same is therefore true of $\frac{\partial \tilde{f}}{\partial \varepsilon}|_{\varepsilon=0} = \frac{\partial \tilde{\eta}}{\partial \varepsilon}|_{\varepsilon=0}$.

Using this, exactly as in (4.1.3) we obtain

$$\left. \frac{d}{d\varepsilon} \right|_0 \tilde{\mathcal{L}}[\tilde{f}] = \int_{\tilde{\Omega}} E_{\tilde{u}}(\tilde{L}) \cdot \left. \frac{\partial \tilde{f}}{\partial \varepsilon} \right|_{\varepsilon=0} d\tilde{x}, \quad (4.2.14)$$

where $E_{\tilde{u}}(\tilde{L})$ is evaluated at $\tilde{u} = \tilde{f}$. To continue, we need to determine $\frac{\partial \tilde{f}}{\partial \varepsilon}$.

To calculate $\tilde{f}(\tilde{x}, \varepsilon)$, we need to solve the first equation in (4.2.13) for x and then insert into the second equation. Thus we have $x = x(\tilde{x}, \varepsilon)$ with

$$\tilde{x} = \Xi(x(\tilde{x}, \varepsilon), f(x(\tilde{x}, \varepsilon)) + \varepsilon \eta(x(\tilde{x}, \varepsilon))).$$

When computing variations of \tilde{L} , the base variables are not allowed to depend on ε , so for $i = 1, \dots, p$ we must have

$$\frac{\partial \tilde{x}^i}{\partial \varepsilon} = 0 = \sum_{j=1}^p D_j \Xi^i \frac{\partial x^j}{\partial \varepsilon} + \sum_{\alpha=1}^q \frac{\partial \Xi^i}{\partial u^\alpha} \eta^\alpha.$$

Using Cramer's rule we can solve for $\frac{\partial x^j}{\partial \varepsilon}$ (and insert $\varepsilon = 0$), to obtain

$$\left. \frac{\partial x^j}{\partial \varepsilon} \right|_{\varepsilon=0} = -\frac{1}{\det J} \sum_{i=1}^p K_{ij} \sum_{\alpha=1}^q \frac{\partial \Xi^i}{\partial u^\alpha} \eta^\alpha,$$

where K_{ij} is the (i, j) -cofactor (i.e., signed minor) of the Jacobian $J(x)$ from (4.2.5). Consequently,

$$\begin{aligned} \left. \frac{\partial \tilde{f}^\beta}{\partial \varepsilon} \right|_{\varepsilon=0} &= \sum_{\alpha=1}^q \frac{\partial \Phi^\beta}{\partial u^\alpha} \eta^\alpha + \sum_{j=1}^p D_j \Phi^\beta \left. \frac{\partial x^j}{\partial \varepsilon} \right|_{\varepsilon=0} \\ &= \frac{1}{\det J} \sum_{\alpha=1}^q \left[\frac{\partial \Phi^\beta}{\partial u^\alpha} \det J - \sum_{i,j=1}^p D_j \Phi^\beta \cdot K_{ij} \frac{\partial \Xi^i}{\partial u^\alpha} \right] \eta^\alpha. \end{aligned}$$

Here, the term in brackets is the expansion of the determinant (4.2.12) along the last column, wherein $\sum_j D_j \Phi^\beta \cdot K_{ij}$ is the row expansion of the $(i, p+1)$ -st minor along its last row (recall that the necessary signs are already contained in the K_{ij}). Thus we have shown that

$$\left. \frac{\partial \tilde{f}^\beta}{\partial \varepsilon} \right|_{\varepsilon=0} = \frac{1}{\det J} \sum_{\alpha=1}^q F_{\alpha\beta} \eta^\alpha.$$

If we insert this into (4.2.14) and change variables we get

$$\left. \frac{d}{d\varepsilon} \right|_0 \tilde{\mathcal{L}}[\tilde{f}] = \int_{\Omega} \left(\sum_{\alpha,\beta=1}^q F_{\alpha\beta} E_{\tilde{u}^\beta}(\tilde{L}) \cdot \eta^\alpha \right) dx,$$

which, on the other hand, must equal

$$\left. \frac{d}{d\varepsilon} \right|_0 \mathcal{L}[f + \varepsilon \eta] = \int_{\Omega} \left(\sum_{\alpha=1}^q E_{u^\alpha}(L) \cdot \eta^\alpha \right) dx.$$

Since η was arbitrary, this proves that (4.2.11) holds at $(x_0, u_0^{(2n)})$. \square

Based on this result, we can now show:

4.2.7 Theorem. *If G is a variational symmetry group of the functional*

$$\mathcal{L}[u] = \int_{\Omega_0} L(x, u^{(n)}) dx,$$

then G is also a symmetry group of the corresponding Euler–Lagrange equations $E(L) = 0$.

Proof. Let $g \in G$ and let f be a solution to the Euler–Lagrange equations. Then since G is a variational symmetry group, (4.2.9) shows that $L(\tilde{x}, \text{pr}^{(n)}\tilde{f}(\tilde{x}))$ and $L(x, \text{pr}^{(n)}f(x))$ are related by the change of variables formula (4.2.4) with $L = \tilde{L}$, where $g^{-1} \cdot (x, u) =: (\Xi(x, u), \Phi(x, u))$. Interchanging the roles of u and \tilde{u} in (4.2.11) we therefore have

$$E_{\tilde{u}^\alpha}(L)(\tilde{x}, \tilde{u}^{(n)}) = \sum_{\beta=1}^q F_{\alpha\beta}(\tilde{x}, \tilde{u}^{(1)}) E_{u^\beta}(L)(x, u^{(n)}), \quad \alpha = 1, \dots, q.$$

Since f is a solution to the Euler–Lagrange equations, $E_{u^\beta}(L)(x, \text{pr}^{(n)}f(x)) = 0$ for all β and all x . Thus also $E_{\tilde{u}^\alpha}(L)(\tilde{x}, \text{pr}^{(n)}\tilde{f}(\tilde{x})) = 0$, so $g \cdot f$ is a solution as well, i.e., G is a symmetry group of $E(L) = 0$. \square

4.3 Conservation laws

4.3.1 Definition. Let $P(x, u^{(n)}) = 0$ be a system of differential equations. A conservation law is a divergence expression

$$\text{Div } F = 0, \tag{4.3.1}$$

which vanishes for all solutions $u = f(x)$ of the system. Here, for some m ,

$$F = (F_1(x, u^{(m)}), \dots, F_p(x, u^{(m)}))$$

is a p -tuple of smooth functions and $\text{Div } F$ is the total divergence from 4.2.2. F then is called a conserved current.

4.3.2 Example. (i) Let $P = \Delta(u)$, where Δ is the Laplace-operator. Then Δ itself is a conservation law because

$$\Delta u = \text{Div}(\text{grad} u) = 0$$

for every solution. Further conservation laws are obtained by multiplying the equation with $u_i = \frac{\partial u}{\partial x^i}$:

$$0 = u_i \Delta u = \sum_{j=1}^p D_j \left(u_i u_j - \frac{1}{2} \delta_i^j \sum_{k=1}^p u_k^2 \right).$$

(ii) If P is any system of ODEs, with independent variable x , then any conservation law is of the form $D_x F = 0$ for all solutions $u = f(x)$ of the system, i.e., $F(x, u^{(n)})$ has to be constant along solutions, so F is a first integral of the system (cf. the remarks following (3.3.3)). In light of this, (4.3.1) can be viewed as a generalization of the concept of a first integral to PDEs.

In applications to physics, often there is a distinguished time-variable t , while the remaining variables (x^1, \dots, x^p) are spatial variables. Then any conservation law takes the form

$$D_t T + \text{Div } X = 0,$$

where Div now is the spatial total divergence (with respect to (x^1, \dots, x^p)). Then T is called the (conserved) *density* and $X = (X_1, \dots, X_p)$ is called the *flux* of the conservation law. They are functions of x, t, u and the derivatives of u with respect to x and t .

Suppose now that $\Omega \subseteq \mathbb{R}^p$ is a spatial domain and let $u = f(x, t)$ be a solution defined for all $x \in \Omega$ and $a \leq t \leq b$. Consider the functional

$$\mathcal{T}_\Omega[f](t) = \int_\Omega T(x, t, \text{pr}^{(m)} f(x, t)) dx, \quad (4.3.2)$$

which, for f , Ω fixed, depends only on t . Then we have:

4.3.3 Proposition. *T and X are the density and the flux of a conservation law of a system of differential equations if and only if, for any bounded domain $\Omega \subseteq \mathbb{R}^p$ with smooth boundary $\partial\Omega$ and for any solution $u = f(x, t)$ defined for $x \in \Omega$ and $t \in [a, b]$ we have*

$$\mathcal{T}_\Omega[f](t) - \mathcal{T}_\Omega[f](a) = - \int_a^t \int_{\partial\Omega} X(x, \tau, \text{pr}^{(m)} f(x, \tau)) \cdot n dS d\tau, \quad (4.3.3)$$

with n the outward-pointing unit normal vector field on $\partial\Omega$.

Proof. Differentiating (4.3.2), by the divergence theorem we have

$$\frac{d}{dt} \mathcal{T}_\Omega[f](t) = \int_\Omega D_t T(x, t, \text{pr}^{(m+1)} f) dx = - \int_{\partial\Omega} X(x, t, \text{pr}^{(m-1)} f) \cdot n dS, \quad (4.3.4)$$

so (4.3.3) follows by integration.

Conversely, differentiating (4.3.3) with respect to t and applying the divergence theorem we obtain

$$\int_\Omega (D_t T(x, t, \text{pr}^{(m+1)} f) + \text{Div } X(x, t, \text{pr}^{(m+1)} f)) dx = 0.$$

As this is to hold for any Ω and f , the claim follows. \square

4.3.4 Example. Consider the motion of an incompressible, inviscid fluid. With $x \in \mathbb{R}^3$ representing the spatial coordinates, let $u = u(x, t) \in \mathbb{R}^3$ denote the velocity of a fluid particle at position x and time t . Let $\rho(x, t)$ be the density and $p(x, t)$ be the pressure of the fluid. We assume that the flow is isentropic (has constant entropy), then p depends only on ρ . The equation of continuity then takes the form

$$\rho_t + \text{Div}(\rho \cdot u) = 0, \quad (4.3.5)$$

where $\text{Div}(\rho u) = \sum_j \frac{\partial(\rho u^j)}{\partial x^j}$ is the spatial total divergence. Also, momentum balance gives the three equations

$$\frac{\partial u^i}{\partial t} + \sum_{j=1}^3 u^j \frac{\partial u^i}{\partial x^j} = - \frac{1}{\rho} \frac{\partial p}{\partial x^i}, \quad i = 1, 2, 3. \quad (4.3.6)$$

Here, the equation of continuity is itself a conservation law with density $T = \rho$ and flux $X = \rho u$. Using (4.3.4) for the functional (4.3.2) corresponding to the flux we get

$$\frac{d}{dt} \int_\Omega \rho dx = - \int_{\partial\Omega} \rho u \cdot n dS.$$

This equation has an immediate physical interpretation: $\int_\Omega \rho dx$ is the total mass of the fluid within Ω , and $\rho u \cdot n$ is the flux of fluid out of a point on the boundary of Ω . This means that the net change of mass inside Ω equals the influx of fluid into Ω via $\partial\Omega$. In particular, if the normal component $u \cdot n$ of the velocity on $\partial\Omega$ vanishes

then there is no change in the mass in Ω and we obtain the law of conservation of mass:

$$\int_{\Omega} \rho \, dx = \text{const.}$$

Furthermore, combining (4.3.5) and (4.3.6) and re-arranging we get three further conservation laws:

$$D_t(\rho u^i) + \sum_{j=1}^3 D_j(\rho u^i u^j + p \delta_i^j) = 0, \quad i = 1, 2, 3.$$

Again using (4.3.4) we obtain the laws of conservation of momentum:

$$\frac{d}{dt} \int_{\Omega} \rho u^i \, dx = - \int_{\partial\Omega} (\rho u^i (u \cdot n) + p n_i) \, dS, \quad i = 1, 2, 3.$$

Here, the first term on the right hand side is the transport of momentum ρu^i due to the flow across the surface $\partial\Omega$, and the second term is the net change in momentum due to the pressure across $\partial\Omega$. Thus $X_j = \rho u^i u^j + p \delta_i^j$ represents the components of the momentum flux.

4.4 Noether's theorem

In this section we prove a fundamental result on the connection between variational symmetries and conservation laws, established by Emmy Noether in 1918. It is of central importance in mathematical physics as the source of many fundamental laws of nature.

4.4.1 Theorem. *Let G be a local one-parameter group of variational symmetries of the variational problem $\mathcal{L}[u] = \int L(x, u^{(n)}) \, dx$. Let*

$$v = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \phi_{\alpha}(x, u) \frac{\partial}{\partial u^{\alpha}} \quad (4.4.1)$$

be the infinitesimal generator of G , and as in (3.10.26) let

$$Q_{\alpha}(x, u^{(1)}) := \phi_{\alpha}(x, u) - \sum_{i=1}^p \xi^i(x, u) u_i^{\alpha}, \quad \alpha = 1, \dots, q.$$

be the characteristic of v . Then there exists a p -tuple

$$F(x, u^{(m)}) = (F_1(x, u^{(m)}), \dots, F_p(x, u^{(m)}))$$

such that

$$\text{Div } F = Q \cdot E(L) = \sum_{\alpha=1}^q Q_{\alpha} \cdot E_{\alpha}(L). \quad (4.4.2)$$

Thus $\text{Div } F = 0$ is a conservation law for the Euler-Lagrange equations $E(L) = 0$.

Proof. Since v is the infinitesimal generator of a variational symmetry, by (4.2.8) and (3.10.29) we have

$$\begin{aligned} 0 &= \text{pr}^{(n)} v(L) + L \text{Div } \xi = \text{pr}^{(n)} v_Q(L) + \sum_{i=1}^p \xi^i D_i L + L \text{Div } \xi \\ &= \text{pr}^{(n)} v_Q(L) + \text{Div } (L \xi), \end{aligned}$$

where $L\xi := (L\xi^1, \dots, L\xi^p)$. We now ‘integrate by parts’ the first term, using (3.10.30):

$$\begin{aligned} \text{pr}^{(n)}v_Q(L) &= \sum_{\alpha, J} D_J Q_\alpha \frac{\partial L}{\partial u_J^\alpha} = \sum_{\alpha, J} Q_\alpha \cdot (-D)_J \frac{\partial L}{\partial u_J^\alpha} + \text{Div } A \\ &= \sum_{\alpha=1}^q Q_\alpha E_\alpha(L) + \text{Div } A, \end{aligned}$$

where $A = (A_1, \dots, A_p)$ is some p -tuple of functions depending on Q , L and their derivatives, but whose concrete form is not important for us. It follows that

$$\text{pr}^{(n)}v_Q(L) = Q \cdot E(L) + \text{Div } A. \quad (4.4.3)$$

Therefore,

$$0 = Q \cdot E(L) + \text{Div } (A + L\xi), \quad (4.4.4)$$

which means that (4.4.2) holds with $F = -(A + L\xi)$. \square

To explicitly calculate F one may in principle follow the calculation of the previous proof. In applications, in most cases one encounters first order variational problems. For these, an explicit formula is given in the following result:

4.4.2 Corollary. *Let $\mathcal{L}[u] = \int L(x, u^{(1)}) dx$ be a first order variational problem, and let v as in (4.4.1) be a variational symmetry. Then*

$$F_i = \sum_{\alpha=1}^q \phi_\alpha \frac{\partial L}{\partial u_i^\alpha} + \xi^i L - \sum_{\alpha=1}^q \sum_{j=1}^p \xi^j u_j^\alpha \frac{\partial L}{\partial u_i^\alpha}, \quad i = 1, \dots, p \quad (4.4.5)$$

form the components of a conservation law $\text{Div } F = 0$ for the corresponding Euler–Lagrange equations $E(L) = 0$.

Proof. Since L depends only on derivatives up to order one, (3.10.30) gives

$$\text{pr}^{(1)}v_Q(L) = \sum_{\alpha=1}^q \left(Q_\alpha \frac{\partial L}{\partial u^\alpha} + \sum_{i=1}^p D_i Q_\alpha \frac{\partial L}{\partial u_i^\alpha} \right).$$

Now setting $A_i := \sum_{\alpha=1}^q Q_\alpha \frac{\partial L}{\partial u_i^\alpha}$ we have

$$\text{Div } A = \sum_{\alpha, i} \left(D_i Q_\alpha \frac{\partial L}{\partial u_i^\alpha} + Q_\alpha D_i \frac{\partial L}{\partial u_i^\alpha} \right).$$

On the other hand, (4.1.4) implies

$$Q \cdot E(L) = \sum_{\alpha=1}^q Q_\alpha \frac{\partial L}{\partial u^\alpha} - \sum_{\alpha=1}^q Q_\alpha D_i \frac{\partial L}{\partial u_i^\alpha},$$

so altogether we obtain

$$\text{pr}^{(1)}v_Q(L) = Q \cdot E(L) + \text{Div } A,$$

verifying (4.4.3). \square

4.4.3 Example. Perhaps the most prominent application of the Noether theorem is to the mechanics of particles. Consider a system of n particles moving in \mathbb{R}^3 in some force field given by a potential. The system is described by the position of its particles, where $\mathbf{x}^\alpha = (x^\alpha, y^\alpha, z^\alpha)$ is the position of the α -th particle, and $\mathbf{x} = (\mathbf{x}^1, \dots, \mathbf{x}^n)$ is a vector containing the information about all positions of the particles. Assuming that the α -th particle has mass m_α , the total kinetic energy of the system is

$$K(\dot{\mathbf{x}}) = \frac{1}{2} \sum_{\alpha=1}^n m_\alpha |\dot{\mathbf{x}}^\alpha|^2.$$

Also, we assume that the force field is determined by a potential $U = U(\mathbf{x}, t)$. Newton's equation of motion then gives

$$m_\alpha \ddot{\mathbf{x}}^\alpha = -\nabla_\alpha U = -(U_{x^\alpha}, U_{y^\alpha}, U_{z^\alpha}), \quad \alpha = 1, \dots, n.$$

The important point to note here is that these equations are the Euler-Lagrange equations of the Lagrange function (or action integral)

$$\mathcal{L}[\mathbf{x}] = \int_{-\infty}^{\infty} (K - U) dt. \quad (4.4.6)$$

Indeed,

$$E_{x^\alpha}(L) = \frac{\partial L}{\partial x^\alpha} - D_t \frac{\partial L}{\partial \dot{x}^\alpha} = -U_{x^\alpha} - m_\alpha \ddot{x}^\alpha,$$

and analogously for y^α and z^α .

By (4.2.8), a vector field

$$v = \tau(t, \mathbf{x}) \frac{\partial}{\partial t} + \sum_{\alpha} \boldsymbol{\xi}^\alpha(t, \mathbf{x}) \cdot \frac{\partial}{\partial \mathbf{x}^\alpha} \equiv \tau \frac{\partial}{\partial t} + \sum_{\alpha} \left(\xi^\alpha \frac{\partial}{\partial x^\alpha} + \eta^\alpha \frac{\partial}{\partial y^\alpha} + \zeta^\alpha \frac{\partial}{\partial z^\alpha} \right)$$

generates a variational symmetry of the Lagrangian (4.4.6) if and only if

$$\text{pr}^{(1)}v(K - U) + (K - U)D_t\tau = 0 \quad \forall(t, \mathbf{x}). \quad (4.4.7)$$

Given a variational symmetry, by Noether's theorem 4.4.2 we obtain a corresponding conservation law (or first integral) with single component (since $p = 1$)

$$\begin{aligned} F &= \sum_{\alpha=1}^n m_\alpha \boldsymbol{\xi}^\alpha \cdot \dot{\mathbf{x}}^\alpha + \tau(K - U) - \sum_{\alpha=1}^n \tau \dot{\mathbf{x}}^\alpha \cdot (m_\alpha \dot{\mathbf{x}}^\alpha) \\ &= \sum_{\alpha=1}^n m_\alpha \boldsymbol{\xi}^\alpha \cdot \dot{\mathbf{x}}^\alpha - \tau(K + U) = \sum_{\alpha=1}^n m_\alpha \boldsymbol{\xi}^\alpha \cdot \dot{\mathbf{x}}^\alpha - \tau E, \end{aligned} \quad (4.4.8)$$

where $E = K + U$ is the total energy. Since $D_t F = 0$ it follows that F has to be constant for any solution of Newton's equations of motion. We now analyze some examples of variational symmetries of (4.4.6) that lead to conservation laws of physical interest.

- $v = \partial_t$: In this case, $\text{pr}^{(1)}v = v$, so (4.4.7) holds if and only if $\partial_t U = 0$, i.e., if and only if U does not explicitly depend on the time t . Since $\tau = 1$ and $\boldsymbol{\xi} = 0$, the conserved quantity given in (4.4.8) then is the total energy E . We obtain that *invariance of a physical system under time translations implies conservation of energy*.

- $v = \sum_{\alpha=1}^n \mathbf{a} \cdot \frac{\partial}{\partial \mathbf{x}^\alpha}$: This is the generator of the translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$, so all particles are simultaneously translated in the same fixed direction $\mathbf{a} \in \mathbb{R}^3$. Also in this case, $\text{pr}^{(1)}v = v$, so (4.4.7) holds if and only if $v(U) = 0$, i.e., if and only if the potential is translationally invariant in the direction \mathbf{a} . As $\tau = 0$ and $\xi^\alpha = \mathbf{a}$ for all α , the corresponding conserved quantity from (4.4.8) is the momentum

$$\sum_{\alpha=1}^n m_\alpha \mathbf{a} \cdot \dot{\mathbf{x}}^\alpha = \text{const.}$$

In particular, if U is invariant under all translations then the total momentum $\sum_{\alpha=1}^n m_\alpha \dot{\mathbf{x}}^\alpha$ is conserved.

- $v = \sum_{\alpha=1}^n (x^\alpha \frac{\partial}{\partial y^\alpha} - y^\alpha \frac{\partial}{\partial x^\alpha})$: This is the generator of a simultaneous rotation of all the masses in the system about some fixed axis, in this case the z -axis. By (3.10.19) we have

$$\text{pr}^{(1)}v = v + \sum_{\alpha=1}^n \left(\dot{x}^\alpha \frac{\partial}{\partial \dot{y}^\alpha} - \dot{y}^\alpha \frac{\partial}{\partial \dot{x}^\alpha} \right).$$

Note that $\text{pr}^{(1)}v(K) = 0$ (since K is invariant under rotations of velocities), and since $\tau = 0$, (4.4.7) shows that v generates a variational symmetry if and only if $v(U) = 0$, i.e., if and only if U is invariant under rotations around the z -axis. In this case, the conserved quantity corresponding to this symmetry via (4.4.8) is the angular momentum around the z -axis

$$\sum_{\alpha=1}^n m_\alpha (x^\alpha \dot{y}^\alpha - y^\alpha \dot{x}^\alpha) = \text{const.}$$

Thus *rotational invariance implies conservation of angular momentum*.

In particular, if we assume that the particles only interact through their mutual gravitational (or electrostatic, ...) attraction, then the potential energy is of the form

$$U(t, \mathbf{x}) = \sum_{\alpha \neq \beta} \gamma_{\alpha\beta} |\mathbf{x}^\alpha - \mathbf{x}^\beta|^{-1}.$$

In this case, all of the above assumptions are satisfied, so we obtain conservation of energy, momentum, and angular momentum.

To conclude this chapter we introduce a straightforward generalization of Noether's theorem that also leads to conservation laws of physical interest.

4.4.4 Definition. Let $\mathcal{L}[u] = \int L(x, u^{(n)}) dx$ be a variational problem. A vector field v on $M \subseteq X \times U$ is called an infinitesimal divergence symmetry of \mathcal{L} if there exists a p -tuple $B(x, u^{(m)}) = (B_1, \dots, B_p)$ of functions of x, u and the derivatives of u such that

$$\text{pr}^{(n)}v(L) + L \cdot \text{Div } \xi = \text{Div } B \quad (4.4.9)$$

for all $(x, u) \in M$.

The point to note is that the Noether theorem 4.4.1 remains valid if in the hypothesis we replace 'variational symmetry' by 'divergence symmetry'. Indeed, the only thing that has to be changed in the proof is that we have to incorporate the new term $\text{Div } B$ into (4.4.4), which now becomes

$$Q \cdot E(L) + \text{Div } (A + L\xi) = \text{Div } B,$$

but we still obtain a conservation law of the form (4.4.2), namely $F = B - A - L\xi$. In particular, in the case of a variational problem of first order, the conserved quantity corresponding to (4.4.5) now becomes

$$F_i = \sum_{\alpha=1}^q \phi_{\alpha} \frac{\partial L}{\partial u_i^{\alpha}} + \xi^i L - \sum_{\alpha=1}^q \sum_{j=1}^p \xi^j u_j^{\alpha} \frac{\partial L}{\partial u_i^{\alpha}} - B_i. \quad i = 1, \dots, p \quad (4.4.10)$$

4.4.5 Example. Returning to the setup of 4.4.3, consider a Galilean boost

$$(t, \mathbf{x}^{\alpha}) \mapsto (t, \mathbf{x}^{\alpha} + \varepsilon t \mathbf{a}),$$

for some $\mathbf{a} \in \mathbb{R}^3$. By (3.10.19), the infinitesimal generator $v = \sum_{\alpha=1}^n t \mathbf{a} \partial_{\mathbf{x}^{\alpha}}$ then has prolongation

$$\text{pr}^{(1)}v = \sum_{\alpha=1}^n \left(t \mathbf{a} \frac{\partial}{\partial \mathbf{x}^{\alpha}} + \mathbf{a} \frac{\partial}{\partial \dot{\mathbf{x}}^{\alpha}} \right),$$

so

$$\text{pr}^{(1)}v(L) = \text{pr}^{(1)}(K - U) = \sum_{\alpha=1}^n m_{\alpha} \mathbf{a} \cdot \dot{\mathbf{x}}^{\alpha} - t \sum_{\alpha=1}^n \mathbf{a} \cdot \nabla_{\alpha} U.$$

Since $\tau = 0$, (4.4.7) demands that this expression should vanish identically, but this is never the case for $\mathbf{a} \neq 0$. However, since

$$\sum_{\alpha=1}^n m_{\alpha} \mathbf{a} \cdot \dot{\mathbf{x}}^{\alpha} = D_t \left(\sum_{\alpha=1}^n m_{\alpha} \mathbf{a} \cdot \mathbf{x}^{\alpha} \right),$$

v is an infinitesimal divergence symmetry if $\mathbf{a} \cdot \nabla_{\alpha} U = 0$, i.e., if U is translationally invariant in the direction \mathbf{a} . The conserved quantity according to (4.4.10) now reads

$$t \sum_{\alpha=1}^n m_{\alpha} \mathbf{a} \cdot \dot{\mathbf{x}}^{\alpha} - \sum_{\alpha=1}^n m_{\alpha} \mathbf{a} \cdot \mathbf{x}^{\alpha}.$$

Here the first sum, when divided by the total mass $\sum_{\alpha} m_{\alpha}$ is the position of the center of mass of the system in the direction \mathbf{a} , and the second is the momentum in that direction. It follows that if U is translationally invariant in a given direction then on the one hand, as shown in 4.4.3, the momentum in that direction is constant. In addition it now follows that the center of mass in that direction is a linear function of t :

$$\text{position of center of mass} = \text{initial position} + t(\text{momentum})/\text{mass}.$$

In particular, if U is invariant under all translations, then the center of mass of any such system moves linearly in one fixed direction.

4.5 Trivial conservation laws and characteristics

There are certain types of conservation laws that do not yield any useful information on the system under consideration and hence are called trivial. A conservation law $F = (F_1, \dots, F_p)$ can be trivial for one of two reasons: triviality of the first kind holds if F itself vanishes on every solution of the system. This kind of triviality is usually easy to eliminate by solving the system itself and its prolongations for certain of the variables u_j^{α} and then substituting for these variables wherever they occur. For example, in the case of an evolution equation $u_t = P(x, u^{(n)})$, any time derivative of u , e.g., u_{tt} , u_{xt} , etc. can be expressed in terms of x , u and spatial derivatives of u .

4.5.1 Example. Consider the system of first order evolution equations

$$u_t = v_x, \quad v_t = u_x,$$

which is equivalent to the one-dimensional wave equation $u_{tt} = u_{xx}$. Then

$$D_t \left(\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 \right) - D_x(u_t u_x) = u_t(u_{tt} - u_{xx}) = 0$$

is a conservation law for this system. As explained above, we can replace the density and the flux of this conservation law by ones that depend only on spatial derivatives. This gives

$$D_t \left(\frac{1}{2} u_x^2 + \frac{1}{2} v_x^2 \right) - D_x(u_x v_x) = 0.$$

The two conservation laws differ by the trivial conservation law

$$D_t \left(\frac{1}{2} u_t^2 - \frac{1}{2} v_x^2 \right) + D_x(v_x u_x - u_t u_x) = 0,$$

for which both density and flux vanish on any solution of the system. In the same way, for any conservation law of an evolution equation there is, up to addition of a trivial conservation law of the first kind, one where density and flux depend only on spatial derivatives.

A conservation law is called trivial of the second kind if the total divergence $\text{Div } F$ in fact vanishes on *all* smooth functions f , whether or not they solve the equation. An example of this kind of triviality is the relation

$$D_x u_y - D_y u_x = 0, \tag{4.5.1}$$

which obviously holds for any smooth function $u = f(x, y)$. Hence (4.5.1) is a trivial conservation law of the second kind for any partial differential equation for functions of x and y . The underlying p -tuples (F_1, \dots, F_p) of conservation laws of the second kind are also called null divergences.

There is in fact a complete characterization of null divergences that is a direct analogue of the characterization of the kernel of the usual divergence operator via the Poincaré lemma. Since the proof of this result, building on the so-called variational complex, is quite involved, we only state the result and refer to [14, Sec. 5.4] for a proof.

4.5.2 Theorem. *Let $F = (F_1, \dots, F_p)$ be a p -tuple of smooth functions depending on $x = (x^1, \dots, x^p)$, (u^1, \dots, u^q) and derivatives of u , and defined on all of $X \times U^{(n)}$. Then the following are equivalent:*

- (i) $\text{Div } F \equiv 0$.
- (ii) *There exist smooth functions G_{jk} , $j, k = 1, \dots, p$, depending on x , u , and derivatives of u , such that $G_{jk} = -G_{kj}$ for all j, k , and*

$$F_j = \sum_{k=1}^p D_k G_{jk}, \quad j = 1, \dots, p \tag{4.5.2}$$

For example, if $p = 3$, then

$$\text{Div } F = D_1 F_1 + D_2 F_2 + D_3 F_3 \equiv 0$$

if and only if F is a ‘total curl’: $F = \text{Curl}(G)$, i.e.,

$$F_1 = D_2G_3 - D_3G_2, \quad F_2 = D_3G_1 - D_1G_3, \quad F_3 = D_1G_2 - D_2G_1$$

(where we have identified $G_{12} = -G_{21}$ from the theorem with G_3 , etc.).

We define a trivial conservation law to be any conservation law that is a linear combination of trivial conservation laws of the first and second kind. Thus $\text{Div } F = 0$ is a trivial conservation law if and only if there exist functions G_{ij} as in 4.5.2 (ii) such that (4.5.2) holds for all solutions of the system. Two conservation laws are called *equivalent* if they differ only by a trivial conservation law. The interesting objects in the study of conservation laws therefore are equivalence classes in this sense.

For the following considerations on characteristics of conservation laws we need some preparations.

4.5.3 Definition. Let $P_\nu(x, u^{(n)}) = 0$, $\nu = 1, \dots, l$, be a system of differential equations, with $P : M^{(n)} \rightarrow \mathbb{R}^l$. The k -th prolongation of this system is the $(n+k)$ -th order system of differential equations

$$P^{(k)}(x, u^{(n+k)}) = 0$$

obtained by differentiating in all possible ways k times. Thus the new system consists of the $\binom{p+k-1}{k} \cdot l$ equations

$$D_J P_\nu(x, u^{(n+k)}) = 0,$$

where $\nu = 1, \dots, l$ and $0 \leq \#J \leq k$.

4.5.4 Example. If P is the heat equation $u_t = u_{xx}$, then the first prolongation $P^{(1)}$ is the system

$$u_t = u_{xx}, \quad u_{xt} = u_{xxx}, \quad u_{tt} = u_{xxt},$$

and the second one contains, in addition, the equations

$$u_{xxt} = u_{xxxx}, \quad u_{xtt} = u_{xxxt}, \quad u_{ttt} = u_{xxtt}.$$

We then clearly have:

4.5.5 Lemma. If $u = f(x)$ is a solution of the system $P(x, u^{(n)}) = 0$, then it is also a solution of every prolongation $P^{(k)}(x, u^{(n+k)}) = 0$, $k = 0, 1, 2, \dots$

4.5.6 Definition. A system of differential equations is called totally nondegenerate if it and all its prolongations are nondegenerate (i.e., of maximal rank and locally solvable).

4.5.7 Lemma. Let $P_\nu(x, u^{(n)}) = 0$, $\nu = 1, \dots, l$, be a totally nondegenerate system of differential equations. Let $Q = Q(x, u^{(m)})$ be a smooth function. Then the following are equivalent:

- (i) Q vanishes for all solutions $u = f(x)$ of the system.
- (ii) There exist differential operators $\mathcal{D}_\nu = \sum_J Q_\nu^J(x, u^{(m)}) D_J$, $\nu = 1, \dots, l$, such that

$$Q = \sum_{\nu=1}^l \mathcal{D}_\nu P_\nu$$

for all $(x, u^{(m)})$.

Proof. (ii) \Rightarrow (i) is obvious.

(i) \Rightarrow (ii): Clearly we can assume that $m \geq n$. By 4.5.5, (i) is equivalent to Q vanishing on all solutions of the prolonged system to $P^{(m-n)}$, which is locally solvable by assumption. Thus it follows that Q vanishes on the zero set of $P^{(m-n)}$. Since, moreover, $P^{(m-n)}$ is nondegenerate, by 3.2.15 there exist smooth functions Q_ν^J ($\#J \leq m-n$, $1 \leq \nu \leq l$) such that

$$Q(x, u^{(m)}) = \sum_J \sum_\nu Q_\nu^J(x, u^{(m)}) D_J P_\nu(x, u^{(m)})$$

□

Let now $\text{Div } F = 0$ be a conservation law of a totally nondegenerate system

$$P(x, u^{(n)}) = 0$$

of differential equations. Then by 4.5.7, $\text{Div } F$ vanishes on every solution of the system if and only if there exist smooth functions $Q_\nu^J(x, u^{(m)})$ such that

$$\text{Div } F = \sum_{\nu, J} Q_\nu^J D_J P_\nu. \quad (4.5.3)$$

We now note that each term in (4.5.3) can be ‘integrated by parts’: for example, if $1 \leq j \leq p$ then

$$Q_\nu^j D_j P_\nu = D_j(Q_\nu^j P_\nu) - D_j(Q_\nu^j) P_\nu.$$

Proceeding in this way, we can cast (4.5.3) in the form

$$\text{Div } F = \text{Div } G + \sum_{\nu=1}^l Q_\nu P_\nu \equiv \text{Div } G + Q \cdot P,$$

where $Q = (Q_1, \dots, Q_l)$ has entries

$$Q_\nu = \sum_J (-D)_J Q_\nu^J, \quad (4.5.4)$$

and all we need to know about $G = (G_1, \dots, G_p)$ is that it depends linearly on the components P_ν of P and their total derivatives. This means that G defines a trivial conservation law of the first kind. Consequently, replacing F by $F - G$ we obtain an equivalent conservation law of the form

$$\text{Div } F = Q \cdot P. \quad (4.5.5)$$

This is called the *characteristic form* of the conservation law (4.5.3), and $Q = (Q_1, \dots, Q_l)$ is called the characteristic of the conservation law.

Unless $l = 1$, the Q_ν from (4.5.5) are in general not uniquely determined. If Q and \tilde{Q} are two l -tuples both satisfying (4.5.5) for the same F , then $(Q - \tilde{Q}) \cdot P = 0$. Since P is nondegenerate, by 3.2.17 it follows from this that $Q - \tilde{Q}$ vanishes on all solutions. Based on this observation, we call a characteristic *trivial* if it vanishes on all solutions of the system, and we call two characteristics equivalent if they differ by a trivial one. In this terminology, characteristics are in general only determined up to equivalence.

4.5.8 Example. To obtain the characteristic form of the conservation law for the heat equation given in 4.5.1, we re-write it in the form (4.5.3):

$$D_t \left(\frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 \right) - D_x(u_t u_x) = u_t D_t(u_t - v_x) + u_t D_x(v_t - u_x).$$

Then (4.5.4) shows that the characteristic is given by

$$Q = (-D_t(u_t), -D_x(u_t)) = (-u_{tt}, -u_{xt}),$$

and an equivalent conservation law in characteristic form can be found through integration by parts:

$$D_t\left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\right) + D_x(-u_t u_x) \cong -u_{tt}(u_t - v_x) - u_{xt}(v_t - u_x).$$

One can show that for systems of differential equations that are totally nondegenerate and normal (i.e., possessing a noncharacteristic direction at every point) the two notions of equivalence we have introduced above actually coincide:

4.5.9 Theorem. *Let $P(x, u^{(n)}) = 0$ be a normal and totally nondegenerate system of differential equations. Let the p -tuples F and \tilde{F} determine conservation laws with characteristics Q and \tilde{Q} , respectively. Then F and \tilde{F} are equivalent as conservation laws if and only if Q and \tilde{Q} are equivalent as characteristics.*

For a proof we refer to [14, Sec. 4.3].

Finally, we may apply the terminology developed in this chapter to re-formulate Noether's theorem 4.4.1 more precisely:

4.5.10 Theorem. *Let G be a local one-parameter group of variational symmetries of the variational problem $\mathcal{L}[u] = \int L(x, u^{(n)}) dx$. Let*

$$v = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

be the infinitesimal generator of G , and as in (3.10.26) let

$$Q_\alpha(x, u^{(1)}) := \phi_\alpha(x, u) - \sum_{i=1}^p \xi^i(x, u) u_i^\alpha, \quad \alpha = 1, \dots, q.$$

be the characteristic of v . Then $Q = (Q_1, \dots, Q_q)$ is also the characteristic of a conservation law for the corresponding Euler–Lagrange equations $E(L) = 0$, i.e., there exists a p -tuple

$$F(x, u^{(m)}) = (F_1(x, u^{(m)}), \dots, F_p(x, u^{(m)}))$$

such that

$$\text{Div } F = Q \cdot E(L) = \sum_{\alpha=1}^q Q_\alpha \cdot E_\alpha(L)$$

is a conservation law in characteristic form for the Euler–Lagrange equations

$$E(L) = 0.$$

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